Magnetoresistance of a two-dimensional electron gas with spatially periodic lateral modulations: Exact consequences of Boltzmann's equation

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On the basis of Boltzmann's equation, and including anisotropic scattering in the collision operator, we investigate the effect of one-dimensional superlattices on two-dimensional electron systems. In addition to superlattices defined by static electric and magnetic fields, we consider mobility superlattices describing a spatially modulated density of scattering centers. We prove that magnetic and electric superlattices in x-direction affect only the resistivity component ρ_{xx} if the mobility is homogeneous, whereas a mobility lattice in x-direction in the absence of electric and magnetic modulations affects only ρ_{yy}. Solving Boltzmann's equation numerically, we calculate the positive magnetoresistance in weak magnetic fields and the Weiss oscillations in stronger fields within a unified approach.

I. INTRODUCTION

Two-dimensional electron systems (2DESs) in high-mobility $Al_{1-x}Ga_xAs$-GaAs heterostructures with one-dimensional lateral superlattices show interesting magnetotransport properties at low temperatures ($T \approx 4$K or less) [1–3]. Experiments on typical samples with electron density $n_{el} \sim 3 \cdot 10^{11}$ cm^{-2}, mobility $\mu \sim 10^6$ cm^2/Vs, and a weak electric modulation [1] in x-direction of period $a \sim 300$ nm revealed the following features. If the homogeneous magnetic field B_0 applied perpendicular to the 2DES is very weak ($B_0 \leq 0.03$T), one finds a pronounced positive magnetoresistance in the component ρ_{xx}. With increasing B_0, this saturates and is followed by the "Weiss oscillations" ($0.1\text{T} \leq B_0 \leq 1\text{T}$), which at higher fields (typically $B_0 \geq 0.6$T at $T = 4.2$K) are superimposed by the Shubnikov-de Haas oscillations. The resistivity component ρ_{yy} does not exhibit such a positive magnetoresistance, but shows Weiss oscillations of opposite phase. No significant effect of the modulation on the Hall resistance has been observed. Experiments on samples with a stronger electric modulation [3,4], and also the recent experiments on magnetically modulated systems [5,6], led to similar results, although they usually did not provide the full information about the conductivity tensor. The Weiss oscillations were first understood within a quantum mechanical picture [7,8]: The modulation lifts the degeneracy of the Landau levels and leads to dispersive Landau bands of oscillatory widths. The group velocity of these bands leads to a "band conductivity", which explains the Weiss oscillations of ρ_{xx}. The oscillatory density of states affects the scattering rate and gives rise to the Weiss oscillations of ρ_{yy} [7,9]. Beenakker [10] pointed out that the Weiss oscillations of ρ_{xx} can also be understood classically in terms of a guiding center drift of cyclotron orbits in the electric modulation field. He proved from Boltzmann's equation that, under the assumption of a constant relaxation time, ρ_{xx} is the only component of the resistivity tensor that is affected by the modulation. The low-B_0 positive magnetoresistance has also been explained classically [4], as being due to channeled trajectories which, in x-direction are localized within a single period of the modulation. Beenakker neglected these channeled orbits in his explicit calculation and could therefore not explain the positive magnetoresistance, whereas Beton *et al.* [4] could not extend their calculation to the regime of Weiss oscillations. So far only the semi-classical approach by Středa and coworkers [11–13] has been applied to both regimes. However, this approach relies on quantum concepts such as energy band structure and magnetic breakdown. A unified calculation, which explains both the low-B_0 magnetoresistance and the Weiss oscillations of ρ_{xx} totally within classical dynamics, is still missing. The purpose of this paper is to present such a classical calculation.

We will extend Beenakker's work [10] by including other modulation sources in addition to a periodic electric field. We will consider a magnetic modulation and also a "mobility modulation", i.e. a periodic position dependence of the collision operator C of Boltzmann's equation, describing a spatial modulation of the density of scatterers. We will also allow for anisotropic scattering of the electrons by, e.g., impurity potentials of finite range, and thus go beyond the relaxation time approximation of Boltzmann's equation. Within our classical approach we will prove that, in the absence of a mobility modulation, a one-dimensional superlattice defined by x-dependent electric and magnetic fields affects only ρ_{xx}, even if anisotropic scattering is included. Allowing for a periodic mobility modulation in x-direction, we can obtain oscillations of ρ_{yy}, however with the same phase as those of ρ_{xx} in the case of a pure electric modulation.

After giving some details about our model in Sect. II, we present the formal calculations revealing the structure of the magnetoresistance tensor in Sect. III and in Appendix A. In Sect. IV we present analytical results: For the regime of Weiss oscillations, we generalize Beenakker's calculation to include magnetic and mobility modulations, and compare with simplified calculations based on the evaluation of the drift velocities of cyclotron

orbits [10,14,15]. For the regime of channeled trajectories we concentrate on the collisionless limit, where we obtain an algebraic dependence of the conductance on the modulation strength. Therefore, in contrast to the regime of Weiss oscillations, a simple power expansion of the transport coefficients with respect to the modulation strength is not possible in this regime.

Numerical results based on a Fourier expansion of Boltzmann's equation are presented in Sect. V. For sufficiently weak modulation strengths, these results cover the whole range of low-field magnetoresistance and Weiss oscillations, and they include the effect of anisotropic scattering as well as of electric, magnetic and mobility modulations. In a final Sect. VI we summarize the most important results.

II. THE MODEL

It is known from quantum mechanical calculations [7,8] that the Weiss oscillations on typical high-mobility samples are observed in a peculiar temperature range. The temperature must be high enough, so that in the regime of Weiss oscillations the Shubnikov-de Hass oscillations are not resolved. On the other hand, it must be low enough, so that the transport coefficients are determined only by the electronic states near the Fermi energy [9,14,15]. To describe the relevant physics in this temperature window, we consider the 2DES in the x-y plane as a degenerate Fermi gas, with Fermi energy $E_F = (m/2)v_F^2$, of (non-interacting) particles with effective mass m and charge $-e$ obeying classical dynamics. The velocity $\mathbf{v} = \dot{\mathbf{r}} = (\dot{x}, \dot{y}, 0)$ of an electron obeys Newton's equation

$$m\dot{\mathbf{v}} = -e\left[\mathbf{F} + (\mathbf{v}\times\mathbf{B})/c\right] . \tag{2.1}$$

To describe a modulated system with period a in equilibrium, we write the electric field as $\mathbf{F}(x) = \nabla V(x)/e$ and the magnetic field as $\mathbf{B}(x) = (0, 0, B_0 + B_m(x))$, and assume that the periodic functions $V(x) = V(x+a)$ and $B_m(x) = B_m(x+a)$ have zero average values. The thermal equilibrium at constant temperature T and electrochemical potential μ^* is described by the distribution function

$$f_{eq}(\mathbf{r},\mathbf{v}) = f_0(E(\mathbf{r},\mathbf{v}); T, \mu^*) , \tag{2.2}$$

where the Fermi function $f_0(E; T, \mu^*) = 1/[1+\exp[(E-\mu^*)/k_BT]]$ depends on the dynamical variables only through the conserved energy

$$E(\mathbf{r},\mathbf{v}) = (m/2)\mathbf{v}^2 + V(x) . \tag{2.3}$$

In more general situations, the distribution function $f(\mathbf{r},\mathbf{v},t)$, which yields electron density n_{el} and current density $\mathbf{j}$ according to

$$n_{el}(\mathbf{r},t) = \langle f\rangle_v , \quad \mathbf{j}(\mathbf{r},t) = \langle -e\mathbf{v}f\rangle_v , \tag{2.4}$$

where $\langle f\rangle_v \equiv [2m^2/(2\pi\hbar)^2]\int d^2v f$, is determined by Boltzmann's equation

$$\left(\frac{\partial}{\partial t} + \dot{\mathbf{r}}\cdot\frac{\partial}{\partial \mathbf{r}} + \dot{\mathbf{v}}\cdot\frac{\partial}{\partial \mathbf{v}}\right) f = C[f; \mathbf{r}, \mathbf{v}, t] \tag{2.5}$$

and suitable boundary conditions. For any reasonable microscopic model of elastic and inelastic scattering [16], the collision operator C vanishes after averaging over the velocity, $\langle C\rangle_v = 0$. As a consequence, the v-integral over Eq. (2.5) yields the continuity equation, $-e\partial n_{el}/\partial t + \nabla\cdot\mathbf{j} = 0$. If we want to choose a phenomenological model for C, we should be careful not to destroy this property.

In the following we consider the stationary, linear response of the modulated system to an external homogeneous electric field $\mathbf{E}^0$, which describes the average effect of a voltage applied across the sample. The result will be a non-zero current density $\mathbf{j}(x)$ and a correction $\delta n_{el}(x)$ to the equilibrium electron density. At low temperatures $[k_BT \ll \mu^* - V(x)]$, the equilibrium density follows from Eqs. (2.2) and (2.4) as $n_{el}(x) = D_0[E_F - V(x)]$, where $D_0 = m/(\pi\hbar^2)$ is the density of states and $E_F = \mu^*(T \to 0)$ the Fermi energy of the homogeneous 2DES.

With the ansatz

$$f_{stat}(\mathbf{r},\mathbf{v}) = f_0(E(\mathbf{r},\mathbf{v}); T, \mu^*) - e\frac{\partial f_0}{\partial \mu^*}\phi(\mathbf{r},\mathbf{v}) \tag{2.6}$$

for the stationary solution, and the assumption that the correction $\phi(\mathbf{r},\mathbf{v})$ to the equilibrium distribution is linear in $\mathbf{E}^0$, we obtain the linearized Boltzmann equation

$$\mathcal{D}\phi(\mathbf{r},\mathbf{v}) - C[\phi; \mathbf{r}, \mathbf{v}] = \mathbf{v}\cdot\mathbf{E}^0 , \tag{2.7}$$

where $\mathcal{D} = \mathbf{v}\cdot\partial/\partial\mathbf{r} + \dot{\mathbf{v}}\cdot\partial/\partial\mathbf{v}$, and $\dot{\mathbf{v}}$ is determined by Eq. (2.1) with the force in the equilibrium state. For $T \to 0$, $\partial f_0/\partial\mu^* \to \delta(E(\mathbf{r},\mathbf{v}) - E_F)$ and with Eq. (2.3), which implies for the magnitude of the velocity $v(x) = v_F[1 - V(x)/E_F]^{1/2}$, the 2D v-integral in Eq. (2.4) reduces to an integral over the polar angle in the velocity space, e.g.

$$\mathbf{j}(x) = e^2 D_0 \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} v(x)\mathbf{u}(\varphi)\phi(x,\varphi) , \tag{2.8}$$

where $\mathbf{u}(\varphi) = (\cos\varphi, \sin\varphi)$, and the dependence of ϕ on E_F is not explicitly indicated. The drift term of Eq. (2.7) reads in these variables

$$\mathcal{D} = v(x)\cos\varphi\frac{\partial}{\partial x} + [\omega_c + \omega_m(x) + \omega_e(x,\varphi)]\frac{\partial}{\partial\varphi} , \tag{2.9}$$

with $\omega_c = eB_0/mc$ the cyclotron frequency due to the average magnetic field, and $\omega_m(x) = eB_m(x)/mc$ and $\omega_e(x,\varphi) = -\sin\varphi\, dv/dx$ are due to the magnetic and the electric modulation, respectively. The collision operator describes the change of the distribution function due to scattering events with a spatial extent which is short

on the scale of the drift motion [16]. If there are different scattering mechanisms, we may write $C[\phi;x,\varphi]=\sum_j C_j[\phi;x,\varphi]$, with

$$C_j[\phi;x,\varphi]=\frac{1}{\tau_j(x)}\int\frac{d\varphi'}{2\pi}P_j(\varphi'-\varphi)[\phi(x,\varphi')-\phi(x,\varphi)]\,, \tag{2.10}$$

where we have expressed the renormalized differential cross sections in terms of relaxation times $\tau_j(x)$ and dimensionless kernels $P_j(\varphi)$. Here, the last term describes the decay of the distribution function due to scattering processes of the j-th type which start in phase space at (x,φ), whereas $\phi(x',\varphi')$ describes the back-scattering into this state. Assuming that the distribution function changes little on the range of the scattering processes, we consider only $x'=x$ in the back-scattering term. For scattering by impurities, the scattering rate is proportional to the impurity density, which may be a function of the position coordinate x in periodically modulated samples. In order to describe a periodic modulation of the electron mobility, we write $1/\tau_j(x)=1/\bar{\tau}_j+r_j(x)$ and assume that the oscillating parts $r_j(x)=r_j(x+a)$ of the scattering rates have zero average values. Further we assume that the kernels $P_j(\varphi)$, with the normalization $\int_{-\pi}^{\pi}d\varphi P_j(\varphi)=2\pi$, have the symmetry $P_j(-\varphi)=P_j(\varphi)$. This follows from the microscopic symmetry if the effect of the magnetic field during an individual scattering process is neglected.

It is interesting to note that the trigonometric functions are eigenfunctions of the collision operator: for $\psi_m(\varphi)=\exp(im\varphi)$ one has

$$C[\psi_m;x,\varphi]=-[1/\tau_{tr}^{(m)}(x)]\,\psi_m(\varphi)\,, \tag{2.11}$$

with the eigenvalues

$$1/\tau_{tr}^{(m)}(x)=\sum_j(1-\gamma_m^{(j)})/\tau_j(x)\,, \tag{2.12}$$

where $\gamma_m^{(j)}=(2\pi)^{-1}\int_{-\pi}^{\pi}d\varphi P_j(\varphi)\cos(m\varphi)$. The same holds for the real part and the imaginary part of ψ_m separately. We will demonstrate in the following that several transport properties do not depend on all the details of the collision operator defined by Eq. (2.10), but only on the transport time $\tau_{tr}(x)\equiv\tau_{tr}^{(1)}(x)$, which we write in the form $1/\tau_{tr}(x)=1/\bar{\tau}_{tr}+r_{tr}(x)$. For instance, the Drude conductivity tensor of the homogeneous 2DES, with $r_{tr}(x)\equiv 0$, depends only on $\bar{\tau}_{tr}$. Thus, the transport scattering rates due to the different scattering mechanisms simply add.

The special case $P_j(\varphi)\equiv 1$ describes *isotropic* scattering, and is identical with the relaxation time approximation, $C_j[\phi]=-(\phi-\phi_{loc})/\tau_j$, where the relaxation is towards the local equilibrium distribution $\phi_{loc}(x)=\int_{-\pi}^{\pi}d\varphi\,\phi(x,\varphi)/2\pi$. This is the correct form of the relaxation time approximation for inhomogeneous systems [17,18]. Neglecting ϕ_{loc} would describe relaxation towards the total equilibrium distribution [cf. Eq. (2.6)], and would violate explicitly the equation of continuity. As we will discuss below, this can lead to qualitatively wrong results for the tranport coefficients.

III. STRUCTURE OF THE RESISTIVITY TENSOR

In this section we prove from Boltzmann's equation that a combined electric and magnetic modulation in x-direction affects only the component ρ_{xx} of the effective resistivity tensor, if there is no position dependence of the scattering rates, $r_j(x)\equiv 0$. All other components then remain those of the unmodulated system. If, on the other hand, there is only a modulation of the scattering rates while $V(x)\equiv 0$ and $B_m(x)\equiv 0$, only ρ_{yy} will be affected. To obtain (within the classical Boltzmann equation approach) a modulation effect on more than one component of the resistivity trensor, one needs a mobility modulation in addition to an electric and/or a magnetic modulation. We will make explicit use of the equation of continuity to derive these results, which generalize an early observation by Beenakker [10] to more general types of modulations and scattering mechanisms. Beenakker solved the problem for a purely electric modulation in the relaxation time approximation. To appreciate the important role of the continuity equation in this context, it is instructive to consider first the simple local limit.

We want to emphasize that we assume translational invariance in the y-direction throughout this work, so that, together with the modulations, current densities and electric field components may depend only on x. Boundary effects are neglected. The experimental situation of a Hall bar with current flow in x-direction will be simulated by assuming that the *average* current density in y-direction vanishes. The experimental boundary conditions of exactly vanishing $j_y(x)$ on the sample boundaries in y-direction would destroy the translational invariance in y-direction and, thus, the simplification resulting from the merely unidirectional modulation. They would require a fully numerical treatment already in the local limit [19], which may become necessary under certain circumstances, but will not be discussed in the present work.

A. Local limit

In the local limit we describe the linear relation between current density $\mathbf{j}(x)$ and driving electric field $\mathbf{F}(x)$ by

$$\hat{\rho}^{\,loc}(x)\,\mathbf{j}(x)=\mathbf{F}(x)\,, \tag{3.1}$$

with a local resistivity tensor $\hat{\rho}^{\,loc}(x)$. It is obtained from the Drude resistivity tensor $\hat{\rho}^{D}$ of the homogeneous system, with components $\rho_{xx}^{D}=\rho_{yy}^{D}=\rho_0$ and

$\rho_{xy}^D = -\rho_{yx}^D = \omega_c\bar{\tau}_{tr}\rho_0$, where $\rho_0 = m/(e^2\bar{\tau}_{tr}\bar{n}_{el})$, by replacing the spatially constant electron density $\bar{n}_{el}$, cyclotron frequency ω_c, and transport time $\bar{\tau}_{tr}$ by their spatially modulated counterparts $n_{el}(x) = \bar{n}_{el}[1-V(x)/E_F]$, $\omega(x) = \omega_c + \omega_m(x)$, and $\tau_{tr}(x) = \bar{\tau}_{tr}/[1 + \bar{\tau}_{tr}r_{tr}(x)]$, respectively. In thermal equilibrium, Eq. (3.1) applies with $\mathbf{j}(x) = 0$ and the vanishing electrochemical field $\mathbf{F}(x)$ given by the sum of the electrostatic field $\nabla V(x)/e$ and the "chemical" contribution $\nabla n_{el}(x)/(eD_0)$. The stationary response to a homogeneous external electric field $\mathbf{E}^0$ is accompanied by a change $\delta n_{el}(x)$ of the electron density, so that $\mathbf{F}(x) = \mathbf{E}^0 + \nabla\delta n_{el}(x)/(eD_0)$ has to be inserted into Eq. (3.1). Since $\delta n_{el}(x)$ must have the periodicity of the electric modulation, we find $\langle\mathbf{F}(x)\rangle = \mathbf{E}^0$, where $\langle ...\rangle$ denotes the average over a period of the modulation. Since current density and field depend only on x, in the stationary state the continuity equation $\nabla\cdot\mathbf{j} = 0$ and Maxwell's equation $\nabla\times\mathbf{F} = 0$ imply that j_x and F_y must be independent of x. With this result, we can define an effective resistivity tensor $\hat{\bar{\rho}}$ by

$$\hat{\bar{\rho}}\,\langle\mathbf{j}(x)\rangle = \mathbf{E}^0\,, \tag{3.2}$$

and calculate it by considering two experimentally relevant situations. First, we assume that the average current density in y-direction is zero. Then, solving the y component of Eq. (3.1) for $j_y(x)$, we obtain

$$\bar{\rho}_{yx} = \frac{F_y}{j_x} = \frac{\langle\rho_{yx}^{loc}/\rho_{yy}^{loc}\rangle}{\langle 1/\rho_{yy}^{loc}\rangle}\,. \tag{3.3}$$

Inserting $j_y(x)$ into the x-component of Eq. (3.1), one obtains

$$\bar{\rho}_{xx} = \frac{\langle F_x\rangle}{j_x} = \left\langle \rho_{xx}^{loc} + [\bar{\rho}_{yx} - \rho_{yx}^{loc}]\,\frac{\rho_{xy}^{loc}}{\rho_{yy}^{loc}}\right\rangle\,. \tag{3.4}$$

In the second, even simpler calculation, we set $j_x = 0$ and obtain

$$\bar{\rho}_{yy} = \frac{F_y}{\langle j_y\rangle} = \frac{1}{\langle 1/\rho_{yy}^{loc}\rangle} \tag{3.5}$$

and

$$\bar{\rho}_{xy} = \frac{\langle F_x\rangle}{\langle j_y\rangle} = \frac{\langle\rho_{xy}^{loc}/\rho_{yy}^{loc}\rangle}{\langle 1/\rho_{yy}^{loc}\rangle}\,. \tag{3.6}$$

Since $\rho_{yx}^{loc} = -\rho_{xy}^{loc}$, one finds from Eqs. (3.3) and (3.6) $\bar{\rho}_{yx} = -\bar{\rho}_{xy}$. If there is no mobility modulation, $r(x) \equiv 0$, one has $\langle 1/\rho_{yy}^{loc}\rangle = \sigma_0$ with $\sigma_0 = 1/\rho_0$ the Drude conductivity of the homogeneous 2DES. Since $\langle\omega(x)\rangle\bar{\tau}_{tr} = \omega_c\bar{\tau}_{tr}$, Eqs. (3.3), (3.6), and (3.5) reduce to the corresponding components of the Drude resistivity for the homogeneous system. Only $\bar{\rho}_{xx} = \langle\rho_{xx}^{loc}(x)\rangle + \bar{\tau}_{tr}^2[\langle\omega(x)^2\rho_{xx}^{loc}(x)\rangle - \omega_c^2/\sigma_0]$ is different from the Drude value $1/\sigma_0$, and shows a positive magnetoresistance ($\propto \omega_c^2$). Note that this result differs from the simple average $\langle\rho_{xx}^{loc}\rangle$, which does not yield any magnetoresistance. If only the mobility is modulated, one has $\langle 1/\rho_{yy}^{loc}\rangle = \sigma_0\,\langle\tau_{tr}(x)\rangle/\bar{\tau}_{tr}$ and only $\bar{\rho}_{yy}$ deviates from the corresponding Drude value for the unmodulated system. If all types of modulation are present, all components of the effective resistivity tensor deviate from the Drude values. For weak modulations, one obtains up to second order in the modulation amplitudes

$$\frac{\bar{\rho}_{xx} - \rho_{xx}^D}{\rho_0} = \left\langle \left(\frac{V}{E_F}\right)^2 + \left(\omega_c\bar{\tau}_{tr}\frac{V}{E_F} + \bar{\tau}_{tr}\omega_m\right)^2\right\rangle + \left\langle \bar{\tau}_{tr}r_{tr}\frac{V}{E_F}\right\rangle\,, \tag{3.7}$$

$$\frac{\bar{\rho}_{xy} - \rho_{xy}^D}{\rho_0} = -\left\langle \bar{\tau}_{tr}r_{tr}\left(\omega_c\bar{\tau}_{tr}\frac{V}{E_F} + \bar{\tau}_{tr}\omega_m\right)\right\rangle\,, \tag{3.8}$$

$$\frac{\bar{\rho}_{yy} - \rho_{yy}^D}{\rho_0} = -\left\langle(\bar{\tau}_{tr}r_{tr})^2\right\rangle - \left\langle \bar{\tau}_{tr}r_{tr}\frac{V}{E_F}\right\rangle\,. \tag{3.9}$$

We want to emphasize that, in order to obtain these results, it was important to satisfy the equation of continuity. If instead we would have taken naively the spatial average of the local conductivity tensor, the resulting effective resistivity tensor would not have the correct structure. A pure electric modulation would have no effect at all. A pure magnetic modulation would affect $\bar{\rho}_{xx}$ and $\bar{\rho}_{yy}$ in the same manner. Although the leading order correction to the xx-component would agree with Eq. (3.7), the corresponding correction to the yy-component would disagree with the correct result (3.9).

B. Nonlocal calculation

The local limit applies to systems in which the electronic mean free path is much shorter than the modulation period. In order to exhibit the Weiss oscillations, the system must be in the opposite limit, with a mean free path considerably larger than the modulation period. For such systems, we consider the current density (2.8) resulting from the solution of Boltzmann's equation (2.7) and define the effective resistivity tensor by Eq. (3.2). We want to emphasize that the spatial average affects only $j_y(x)$, whereas j_x must be independent of x by virtue of the continuity equation.

By formal, but exact manipulations of the Boltzmann equation we show in Appendix A that the tensor inverse of $\hat{\bar{\rho}}$ can be written as

$$\hat{\bar{\sigma}} = \hat{\sigma}^D + \rho_0\,\hat{\sigma}^D\hat{\mathbf{\Gamma}}\,\hat{\sigma}^D \tag{3.10}$$

where $\hat{\sigma}^D$ is the Drude conductivity tensor of the corresponding homogeneous system, and $\hat{\mathbf{\Gamma}}$ has the components

$$\begin{aligned}\Gamma_{xx} &= -\langle\bar{\tau}_{tr}r_{tr}V/E_F\rangle - G_{xx}\,, \quad \Gamma_{xy} = -G_{xy}\\ \Gamma_{yy} &= +\langle\bar{\tau}_{tr}r_{tr}V/E_F\,\rangle + G_{yy}, \quad \Gamma_{yx} = +G_{yx}\,,\end{aligned} \tag{3.11}$$

with

$$G_{\mu\nu} = \frac{2\bar{\tau}_{tr}}{v_F^2} \left\langle \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi}\, g_\mu \chi_\nu \right\rangle . \tag{3.12}$$

Here

$$g_x = \frac{v_F^2}{2} \frac{dV/dx}{E_F} + \omega_m v_y\,, \quad g_y = -r_{tr} v_y\,, \tag{3.13}$$

and χ_ν is the solution of the modified Boltzmann equation

$$[\mathcal{D} - C]\, \chi_\nu = g_\nu\,. \tag{3.14}$$

Then, by tensor inversion, $\hat{\bar{\sigma}}^{-1} = \hat{\bar{\rho}} = \hat{\rho}^D + \Delta\hat{\rho}$, we obtain from Eq. (3.10)

$$\Delta\hat{\rho}/\rho_0 = -\hat{\mathbf{\Gamma}} \left[\hat{\mathbf{1}} + \rho_0\, \hat{\sigma}^D \hat{\mathbf{\Gamma}} \right]^{-1} . \tag{3.15}$$

Explicit matrix inversion and multiplication finally leads to the relatively simple result

$$\frac{\Delta\hat{\rho}}{\rho_0} = -\left[\hat{\mathbf{\Gamma}} + \frac{\det\hat{\mathbf{\Gamma}}}{1 + \omega_c^2\, \bar{\tau}_{tr}^2} \frac{\hat{\rho}^D}{\rho_0} \right] \frac{1}{D}\,, \tag{3.16}$$

with $\det\hat{\mathbf{\Gamma}} = \Gamma_{xx}\Gamma_{yy} - \Gamma_{xy}\Gamma_{yx}$ and

$$D = 1 + \frac{\Gamma_{xx} + \Gamma_{yy} + \omega_c \bar{\tau}_{tr}\, (\Gamma_{xy} - \Gamma_{yx}) + \det\hat{\mathbf{\Gamma}}}{1 + \omega_c^2\, \bar{\tau}_{tr}^2}\,. \tag{3.17}$$

If Γ_{xx} or Γ_{yy} is the only nonzero matrix element of $\hat{\mathbf{\Gamma}}$, Eq. (3.16) reduces to (A16) or (A17), respectively, i.e., either only $\Delta\rho_{xx}$ or only $\Delta\rho_{yy}$ is nonzero, and we obtain the same tensor structure as in the local limit. As shown in Appendix A, these results can be derived without the explicit formal result (3.16).

We want to emphasize that only the global transport time $\bar{\tau}_{tr}$, defined below Eq. (2.12), enters these considerations about the structure of the effective resistivity tensor, not any further details of the scattering mechanisms. The latter enter, of course, the solutions of Eq. (3.14) and determine, according to (3.12), the numerical values of the tensor components of $\hat{\mathbf{\Gamma}}$.

IV. ANALYTIC RESULTS

A. Weiss oscillations

If the modulation is weak, in the sense that the forces due to the electric and the magnetic modulations are weak as compared to the Lorentz force due to the average magnetic field B_0, a perturbation expansion with respect to the amplitudes of the modulation is possible. To obtain the conductivity correct to second order, we need, according to Eq. (3.12), χ_ν to first order, and we can replace the square bracket in Eq. (3.15) by unity. Since the right hand side of Eq. (3.14) is already of first order, we should neglect modulation effects on C and on $\mathcal{D}$, i.e. replace in Eq. (2.9) $v(x)$ with v_F and omit $\omega_m(x)$ and $\omega_e(x,\varphi)$. In this approximation the different Fourier coefficients of the expansions $V(x) = \sum_{q\neq 0} V_q \exp(iqx)$, and similar for $\omega_m(x)$ and $r_{tr}(x)$ with coefficients ω_q and r_q, respectively, do not mix in Eq. (3.14) and lead to uncoupled integro-differential equations of the variable φ. We now follow Beenakker [10] and consider only isotropic scattering, i.e. we take $\gamma_0^{(j)} = 1$ and $\gamma_m^{(j)} = 0$ for $m \neq 0$ in Eqs. (2.11), and define the relaxation time $\tau \equiv \bar{\tau}$ by the spatial average $1/\tau = \langle 1/\tau(x) \rangle$. Then we can solve these equations analytically and obtain

$$G_{xx} = \frac{1}{2} \sum_{q\neq 0} \left\{ |\lambda q V_q/E_F|^2 Q_q^{ee} \right. \tag{4.1}$$
$$\left. - \left[4\lambda q \tau \omega_q V_{-q}/E_F \right] Q_q^{em} + |2\tau\omega_q|^2 Q_q^{mm} \right\} ,$$

$$G_{xy} = \sum_{q\neq 0} \tau r_{-q} [(\lambda q V_q/E_F) Q_q^{em} - 2\tau\omega_q Q_q^{mm}]\,, \tag{4.2}$$

$$G_{yx} = -G_{xy}\,, \qquad G_{yy} = 2 \sum_{q\neq 0} |\tau r_q|^2 Q_q^{mm}\,, \tag{4.3}$$

with $Q_q^{ee} = S_q^{(0,0)}/N_q$, $Q_q^{em} = S_q^{(0,1)}/N_q$, and $Q_q^{mm} = S_q^{(1,1)} + (S_q^{(0,1)})^2/N_q$, where $N_q = 1 - S_q^{(0,0)}$ and

$$S_q^{(\mu,\nu)} = \sum_{m=-\infty}^{\infty} \frac{J_m^{(\mu)}(Rq)\, J_m^{(\nu)}(Rq)}{1 + (m\omega_c\tau)^2}\,, \tag{4.4}$$

where $\lambda = v_F\tau$ is the mean free path, $R = v_F/\omega_c$ is the cyclotron radius, and $J_m^{(0)}(z) = J_m(z)$ and $J_m^{(1)}(z) = J_m'(z)$ are the Bessel functions and their derivatives. For $\omega_q \equiv 0$, $r_q \equiv 0$, and $V_q = \delta_{|q|,2\pi/a} V_{rms}/\sqrt{2}$ this reduces to Beenakker's result [10]. The term $S_q^{(0,0)}$ in the denominator N_q arises from the back-scattering term in the collision operator, which describes relaxation towards the local equilibrium. It is thus a consequence of the equation of continuity.

1. *Clean limit, $\omega_c\tau \gg 1$*

In the limit $\omega_c\tau \gg 1$, only the term with $m = 0$ survives in the sum of Eq. (4.4), and in Eq. (A16) the term $\tilde{\Gamma}_{xx}$ may be neglected, so that $\Delta\rho_{xx}/\rho_0 \approx G_{xx}$. Then, rearranging terms with q and $-q$, the curly bracket in Eq. (4.1) can be replaced by $|\lambda q \mathcal{S}_q/E_F|^2$ with $\mathcal{S}_q = V_q J_0(Rq) + (k_F/q)\hbar\omega_q J_1(Rq)$,

$$\Delta\rho_{xx}/\rho_0 \approx \frac{1}{2} \sum_{q\neq 0} \frac{|\lambda q \mathcal{S}_q/E_F|^2}{1 - J_0^2(Rq)}\,. \tag{4.5}$$

This result is, apart from the denominator $1 - J_0^2(Rq)$, in agreement with a recent simplified calculation [15] based on an evaluation of Chambers' formula for the conductivity in terms of the drift velocity of cyclotron orbits.

That calculation corresponds to a naive relaxation time approximation in Boltzmann's equation with relaxation towards the total instead of the local equilibrium distribution function, and thus violates the equation of continuity for the modulated system. Indeed, the nontrivial denominator in the present calculation results from the back-scattering term in Eq. (2.10).

In Fig. 1 we demonstrate the range of validity of several approximations to the results (4.1) - (4.4) for the special case of a simply harmonic electrical modulation $V(x) = V_0 \cos qx$. Figure 1(a) shows, for several values of the mean free path λ, the quantity $S_q^{(0,0)}$ of Eq. (4.4), which determines the denominator of the magnetoresistance corrections. Figures 1(b) and (c) show results for the magnetoresistance, calculated from Eq. (4.1), with different normalizations which are suitable to discuss the local limit, (b), and the limit of infinite λ, (c), respectively. The curves are plotted versus the inverse cyclotron radius, which is proportional to the average magnetic field B_0. In the limit $\lambda \to \infty$, $S_q^{(0,0)} = J_0^2(Rq) \approx (2/\pi Rq)\cos^2(Rq - \pi/4)$ decreases with decreasing B_0, with an infinite number of zeroes and with values at the maxima, which approach zero linearly with B_0. The same then holds for the magnetoresistivity curve obtained from Eq. (4.5) and for the thin line in Fig. 1 (c), which corresponds to the simplified result of Refs. [14,15] and is calculated from Eq. (4.5) by replacing the denominator by 1. For $1/Rq < 0.1$ the denominator of Eq. (4.5) is close to unity ($S_q^{(0,0)} < 0.1$), and the thin line approximates the result of Eq. (4.5) quite well. At higher magnetic fields, $1/Rq > 0.2$, the result of Eq. (4.5) agrees well with the magnetoresistivity curve obtained for $\lambda q = 30$ in Fig. 1 (c). For a sample with a given value of the mean free path, Eq. (4.5) does not hold uniformly for all values of the average magnetic field. With decreasing B_0 values, $\lambda q/Rq = \omega_c\tau$ becomes smaller and more and more terms contribute to the sum in Eq. (4.4). As a consequence, the minima of the Weiss oscillations of $S_q^{(0,0)}$ and $\Delta\rho_{xx}$ are risen to positive values, the amplitudes of the oscillations are increasingly suppressed with decreasing B_0 values, and, in the limit $B_0 \to 0$, a finite positive value is approached with zero slope.

Whereas $S_q^{(0,0)}$ in the limit of large B_0 ($Rq \to 0$) approaches 1 for all values of λ [since $J_0(0) = 1$ and $J_m(0) = 0$ for $m \neq 0$], it shows an overall increase with decreasing λ for all values of Rq, and approaches 1 uniformly in B_0 for $\lambda q \to 0$ [since $\sum_{-\infty}^{\infty} J_m^2(z) \equiv 1$]. Thus, the denominator N_q in Eqs. (4.1)-(4.3) becomes most important in the local limit $\lambda q \to 0$ and in the "quasi-local" limit $Rq < 1$, where the cyclotron radius R becomes smaller than the period $a = 2\pi/q$ of the modulation. Figure 1 (b) shows the magnetoresistivity data for $\lambda q = 9$ and $\lambda q = 4$ in a representation which is more convenient for the discussion of the local limit $\lambda q \to 0$. For $\lambda q = 1$, the corresponding result approximates already closely the local result of Eq. (3.4), $\Delta\rho_{xx}/\rho_0 = (1/2)(V_0/E_F)^2\,[1 + (\lambda/R)^2]$, which is indicated by the dash-dotted line.

In the "quasi-local" high magnetic field limit, $Rq \ll 1$, the denominator becomes $1 - J_0^2(Rq) \approx (Rq)^2/2 \ll 1$, so that, for $V(x) = V_0 \cos qx$, we obtain a quadratic magnetoresistance,

$$\Delta\rho_{xx}/\rho_0 \approx (\omega_c\tau)^2(V_0/E_F)^2/2\,, \quad (Rq \ll 1)\,, \tag{4.6}$$

whereas the calculation of Ref. [15] yields a saturation for large ω_c, $\Delta\rho_{xx}^{[15]}/\rho_0 \approx (\lambda q)^2(V_0/E_F)^2/4$. Experiments seem to show a quadratic magnetoresistance at high magnetic fields, although masked by the onset of Shubnikov-de Haas oscillations. For a sinusoidal magnetic modulation, $\omega_m(x) = \omega_m^0 \cos qx$, we obtain from Eq. (4.5) at large B_0 values a saturation,

$$\Delta\rho_{xx}/\rho_0 \approx (\omega_m^0\tau)^2/2\,, \quad (Rq \ll 1)\,, \tag{4.7}$$

whereas Ref. [15] yields $\Delta\rho_{xx}^{[15]}/\rho_0 \approx (\lambda q)^2(\omega_m^0/\omega_c)^2/4$, which becomes small $\propto B_0^{-2}$. We want to emphasize that, in this "quasi-local" limit, the B_0 dependence of the magnetoresistance obtained from Eq. (4.5) agrees with that of the local approximation (3.7).

Figure 2 shows the prediction of Eqs. (4.1) - (4.4) for the case when two types of modulations are simultaneously present. In Fig. 2 (a) an electric and a magnetic cosine modulation of comparable effective strengths are considered. The dash-dotted line refers to a pure electric and the solid line to a pure magnetic modulation, whereas the dashed lines refer to a combination of both. For fixed amplitudes of the electric and the magnetic cosine modulations, the resulting magnetoresistivity curve depends strongly on the phase difference between both. The long-dashed line is Fig. 2 (a) is obtained for in-phase modulations, and the short-dashed line for a phase shift of a half period. A systematic investigation of the interference effects occurring due to the superpositon of an electric and a magnetic modulation has been given within the simplified treatment of Ref. [15]. Since for $\lambda q > 20$ this treatment is qualitatively correct in the regime of Weiss oscillations, we will not repeat this discussion here.

Figure 2 (b) shows the components of the magnetoresistivity tensor for a combination of electric and mobility modulations. We assume here in-phase cosine modulations for the electrostatic potential energy and the scattering rate of the electrons, and comment on opposite phases below. According to Eqs. (4.1) - (4.4), $\Delta\rho_{xx}$ is determined by the quantity Q_q^{ee}, $\Delta\rho_{yy}$ by Q_q^{mm}, and $\Delta\rho_{xy} = -\Delta\rho_{yx}$ by Q_q^{em}, all of which contain different combinations of Bessel functions and their derivatives. For $\lambda q > 20$, in the regime of Weiss oscillations Q_q^{ee} is proportional to the square of $J_0(Rq)$ and Q_q^{mm}, which also determines the results of a pure magnetic modulation, is proportional to the square of the derivative $J_0'(Rq)$. This leads to antiphase oscillations of Q_q^{ee} and Q_q^{mm} as depicted in Fig. 2 (a) by the dash-dotted and the solid lines, respectively. However, according to Eqs. (3.11),

(3.15), and (4.1) - (4.3), $\Delta\rho_{xx}$ and $\Delta\rho_{yy}$ exhibit *in-phase oscillations* since, apart from B_0-independent offsets $+\tau r_m^0(V_0/E_F)/2$ and $-\tau r_m^0(V_0/E_F)/2$, respectively, $\Delta\rho_{xx}$ is proportional to Q_q^{ee}, whereas $\Delta\rho_{yy}$ is proportional to $-Q_q^{mm}$. Thus, a mobility modulation can not explain the *anti-phase oscillations* of $\Delta\rho_{yy}$ observed in the early experiments [1], as mentioned in the Introduction. In the situation considered in Fig. 2, Q_q^{em} is approximately given by the derivative of Q_q^{ee} with respect to Rq. Thus in Fig. 2 (b), where the resistivity components are plotted versus $1/Rq$, $\Delta\rho_{yx}$ appears as the derivative of $\Delta\rho_{xx}$. If we change the relative sign of electric and mobility modulation, the sign of $\Delta\rho_{yx}$ will also change, whereas the phase relation between $\Delta\rho_{xx}$ and $\Delta\rho_{yy}$ remains unchanged.

We do not show a figure analog to Fig. 2 (b) for a superposition of a magnetic and a mobility modulation, since in this case all components of the magnetoresistivity tensor are determined by Q_q^{mm}. As a consequence, $\Delta\rho_{yx}$ would show the same appearance as $\Delta\rho_{xx}$, i.e. as the solid line in Fig. 2 (a), and $\Delta\rho_{yy}$ would appear as $-\Delta\rho_{xx}$, i.e. with the opposite phase. In contrast to the case of electric and mobility modulation, in the case of magnetic and mobility modulation there is no constant offset in the diagonal components of the resistivity tensor, and $\Delta\rho_{yx}$ does not assume negative values.

2. Damping of Weiss oscillations

The value of $S_q^{(0,0)}$ for weak magnetic fields, $Rq \geq 2$, depends strongly on λq, and Weiss oscillations are washed out for $1/Rq < 0.5/\lambda q$, i.e., for $\omega_c\tau < 0.5$. If the mean free path is too small, $\lambda q \leq 2$ (here we have in mind that the period of the modulations, and thus q is fixed), no oscillations survive, since too many Bessel functions contribute to the sum in Eq. (4.4) ($\omega_c\tau < 1$ in the $1/Rq$ regime where oscillations can be expected). In this regime the calculation of Ref. [15] is not applicable, since it assumes $\lambda \gg R$.

We can also re-obtain the local limit $\lambda q \to 0$ from Eqs. (4.1) - (4.3), if we expand the Bessel functions in Eq. (4.4) for small arguments, $Rq \to 0$, and keep only the leading orders. This means that we consider the limit of large modulation period, but make no assumption about the magnitude of $\omega_c\tau$. In this limit, only the terms with $m = 0$ and $m = \pm 1$ contribute to Eq. (4.4), and we exactly recover Eqs. (3.7) - (3.9), with τ instead of $\bar{\tau}_{tr}$. To obtain this result correctly, it is essential to calculate the denominator (which vanishes in this limit) consistently up to the order $(Rq)^2$, which includes contributions from the Bessel functions J_0 and J_1. This demonstrates nicely that the denominator in Eqs. (4.1) - (4.3) indeed is important in order to obtain the correct local-limit results, Eqs. (3.7) - (3.9), which were derived by exploiting explicitly the equation of continuity.

B. Low-field magnetoresistance

If the average magnetic field B_0 becomes too small, the analytic solution of Eq. (3.14) for small modulation amplitudes is no longer applicable. It was obtained under the assumption, that for the calculation of χ_ν the modulation contributions $\omega_m(x)$ and $\omega_e(x,\varphi)$ to the drift operator (2.9) could be neglected as compared to the average cyclotron frequency ω_c. Physically this means that the modulations modify the cyclotron motion in the average magnetic field only weakly and lead essentially to a drift of the cyclotron orbits. If however, for a given modulation, B_0 becomes smaller than a critical B_{crit} (or $R > R_{crit}$), this is no longer true and other types of orbits ("channeled orbits") occur which are confined in the x-direction to a single period of the modulation superlattice. The effect of these orbits, which lead to a positive magnetoresistance at small B_0 [4], is not included in Eqs. (4.1) - (4.3).

1. Classification of orbits

To characterize the channeled orbits and to elucidate their importance for the magnetoresistance, we proceed as follows. We note that, for a one-dimensional modulation in x-direction, there exists a second conserved quantity in addition to the energy, Eq. (2.3). Integrating the y-component of Newton's equation (2.1), we can write

$$v_y - \frac{e}{mc}\left[xB_0 + \int_0^x dx' B_m(x')\right] = \frac{p_y}{m}\,, \qquad (4.8)$$

where p_y is a constant of motion, which has the meaning of a canonical momentum. Solving for v_y and inserting this into Eq. (2.3), we obtain for the projection of the orbit on the x-direction

$$(m/2)v_x^2 + \tilde{V}(x) = E_F\,, \qquad (4.9)$$

with the effective one-dimensional potential

$$\tilde{V}(x) = V(x) + \frac{m}{2}\omega_c^2\left[x - x_0 + \int_0^x dx' \frac{B_m(x')}{B_0}\right]^2. \qquad (4.10)$$

Here we assume $B_0 \neq 0$, and the constant of motion is written as $x_0 = -cp_y/eB_0$, which in the absence of modulations is the x coordinate of the center of the cyclotron orbit. For each value of x_0 one obtains a different location of the effective potential $\tilde{V}(x)$ on the x-axis, and thus a different orbit located between turning points, at which $v_x = 0$. If x is a turning point, $\tilde{V}(x) = E_F$, Eq. (4.10) yields two possible values for the constant x_0:

$$x_0^{\pm}(x) = x + \int_0^x dx' \frac{B_m(x')}{B_0} \pm R\sqrt{1 - \frac{V(x)}{E_F}}\,. \qquad (4.11)$$

Since $\tilde{V}(x) \leq E_F$ holds for any orbit passing through a given position x, only orbits with x_0 values in the interval $x_0^-(x) \leq x_0 \leq x_0^+(x)$ can pass through x. (We specify orbits only up to an arbitrary shift in y direction.) On the other hand, a given value of x_0 determines an orbit between turning points x_t satisfying $x_0^-(x_t) = x_0$ or $x_0^+(x_t) = x_0$. Thus the curves $x_0 = x_0^\pm(x)$ yield the constant of motion of the orbits which have x as a turning point. We use this property to distinguish two types of orbits.

Orbits of the first type we call "drifting orbits". These have their left turning point x_l satifying $x_0^+(x_l) = x_0$ and their right turning point x_r satifying $x_0^-(x_r) = x_0$. For sufficiently smooth modulation (cf. Appendix B) and large B_0 this is the only type of orbits, as we demonstrate in the left panels of Fig. 3 for an electric cosine modulation. In the limit of vanishing modulation these "drifting orbits" reduce to cyclotron orbits, i.e., to circles, and $x_r - x_l \to 2R$. The effect of the modulation in x-direction is to deform the circular cyclotron orbits and to superimpose on the cyclotron motion a drift velocity in y-direction. The drift vecolity depends on the position of the orbit in the modulation field and may vanish for highly symmetric positions. In the left panel of Fig. 3 one orbit is nearly symmetric with respect to the potential maximum at $x = 0$, another one nearly symmetric about the minimum at $x = 1.5a$. The small distance between neighboring loops of the orbits (three loops are shown) indicates a low drift velocity.

Orbits of the second type, which we call "channeled orbits", occur if the curves $x_0^\pm(x)$ have local extrema, i.e., for sufficiently strong modulation or weak magnetic field B_0. For "channeled orbits" we require that their turning points both satisfy either $x_0^+(x_l) = x_0^+(x_r) = x_0$ or $x_0^-(x_l) = x_0^-(x_r) = x_0$, i.e. in plots like those in the upper panels of Fig. 3 both turning points are located either on $x_0^+(x)$ or on $x_0^-(x)$. These channeled orbits (see middle and right panels of Fig. 3) are confined to a single period a of the modulation and have no counterparts in the unmodulated system. For an electric modulation they occur near potential minima. For a magnetic modulation they occur where the total magnetic field $B_0 + B_m(x)$ changes sign [21].

Each value x_0 of the constant of motion determines uniquely a single drifting orbit (of course, modulo a shift in y-direction). Channeled orbits may occur with the same x_0 value, however only in intervals of the x axis which are not entered by that drifting orbit. At a given position x, each allowed x_0 value characterizes uniquely a single orbit through x. In the limit $B_0 = 0$, drifting orbits degenerate into wavy trajectories with periodic velocity $\mathbf{v}(x + a) = \mathbf{v}(x)$ and nonzero mean velocity $\langle v_x \rangle$ in x-direction. The mean value in y-direction is $\langle v_y \rangle = p_y/m$ (note that $x_0/R = -p_y/mv_F$, and for pure electric modulation $v_y = p_y/m$ is constant). In this limit two trajectories with $\langle v_x \rangle > 0$ and $\langle v_x \rangle < 0$ occur with the same $\langle v_y \rangle$, as indicated by the dashed lines in the lower right panel of Fig. 3.

If x_0 coincides with the value of a local maximum of $x_0^-(x)$ or a local minimum of $x_0^+(x)$, the orbits near the corresponding turning point x_t become critical in the sense that they do not reach this turning point, but approach in the $x-y$ plane the straight line $x = x_t$ asymptotically.

2. Effect of channeled orbits

As has been emphasized by Beton et *al.* [4], the channeled orbits are responsible for the strong positive magnetoresistance of modulated systems in weak perpendicular magnetic fields. In Appendix B we calculate the fraction $f_{co}(B_0, V_0)$ of the phase space covered by channeled orbits, as a function of the (average) magnetic field B_0 and the modulation strengths V_0, for two electrostatic modulation models. For the smooth cosine modulation with a fixed V_0, channeled orbits exist only at sufficiently low $B_0 < B_{crit}(V_0)$, whereas for a discontinuous step modulation skipping orbits survive at arbitrarily high magnetic fields. For both models the fraction of channeled orbits at zero magnetic field, $f_{co}(0, V_0)$, increases at low modulation strengths proportional to $|V_0/E_F|^{1/2}$. We also sketch, for $B_0 = 0$ and in the nonlocal limit of large mean free path, a calculation in the spirit of Ref. [15] which shows that the resistivity correction increases as $|V_0/E_F|^{3/2}$ with the modulation amplitude. This indicates that, for $B_0 = 0$ and weak modulation, no expansion of the transport coefficients in a power series of V_0 is possible. The same is true at finite magnetic field if channeled orbits exist. Indeed, for the step model which allows for channeled orbits at arbitrary strong B_0, the sum over the Fourier coefficients in the second-order formula (4.1) for the magnetoconductivity diverges for all values of B_0, indicating that such an expansion does not exist [14].

To include the effect of channeled orbits on the magnetoresistance properly, we have solved Boltzmann's equation numerically. The results show in the limit of long mean free path and zero magnetic field the same $|V_0/E_F|^{3/2}$ behavior as obtained analytically in the simplified approach, and will be presented in the next section.

V. NUMERICAL RESULTS

A. Some technical details

In order to cover the whole magnetic field range, we have solved Eq. (3.14) numerically, using Fourier expansions for the dependences on both x and φ. According to Eq. (2.11), the collision operator, Eq. (2.10), is diagonal in the Fourier representation. The drift term, Eq. (2.9), becomes simple for strictly harmonic modulations, provided that the electric modulation is weak

enough and $v(x)$ can be approximated by v_F. Then the drift term couples only neighboring Fourier coefficients, and Eq. (3.14) leads to an infinite set of linear equations for the Fourier coefficients with a sparsely occupied matrix. Truncating the infinite set by considering only the lowest N_x and N_φ Fourier coefficients for the x- and the φ-dependence, respectively, we could exploit the sparse nature of the system matrix and handle approximations of the infinite set with large dimensions $N_x \cdot N_\varphi$. The number of Fourier coefficients needed to obtain converged results increases with increasing modulation strenghts and with increasing mean free path. The convergence in the regime of Weiss oscillations is much better than in the low-B_0 regime, where the channeled orbits are important. For the pure electric cosine modulation with $V_0/E_F = 0.02$ we needed $N_x = 13$ and $N_\varphi = 550$ to achieve convergence in the low-magnetic-field regime $5 \cdot 10^{-5} < 1/qR < 0.02$, whereas $N_x = 4$ and $N_\varphi = 450$ was sufficient for $1/qR > 0.5$. The reason for the poor convergence at low B_0 is that the distribution function calculated from Eq. (3.14) developes very sharp and rapidly changing structures near the angles $\varphi = \pi/2$ and $3\pi/2$, originating from the turning points of the channeled orbits. Owing to these numerical problems, our Fourier expansion approach is reliable only for weak modulations ($V_0/E_F \leq 0.2$).

B. Relaxation time approximation

A typical low-magnetic-field result for the pure electric modulation $V(x) = V_0 \cos qx$ with $V_0/E_F = 0.02$ is shown in Fig. 4. The solid line is the result of our numerical calculation. The dotted line shows the result of the analytical approximation (4.1). The dash-dotted line indicates the contribution of the channeled orbits and is obtained as follows [4]. For weak modulation and $\omega_c\tau \gg 1$, the correction to the Drude resistivity tensor can be written as $\Delta\rho_{xx}/\rho_0 \approx \omega_c^2\tau^2\Delta\sigma_{yy}/\sigma_0$. The contribution of channeled orbits may be estimated by their mean square drift velocity $\Delta\sigma_{yy}/\sigma_0 = \langle v_d^2\rangle/(v_F^2/2)$, cf. Eq. (B12). Approximating $v_d^2 \approx v_F^2$, this becomes $\approx 2f_{co}^{\cos}(B_0, V_0)$, i.e., twice the fraction of channeled orbits at the Fermi energy (see appendix B). The dash-dotted line in Fig. 4 represents $\Delta\rho_{xx}/\rho_0 \approx 2\omega_c^2\tau^2 f_{co}^{\cos}(B_0, V_0)$ with an additional vertical shift to match our numerical result at $B_0 = 0$. The vertical straight line indicates the critical magnetic field $B_{crit}^{Beton} = cqV_0/ev_F$ at which the channeled orbits die out [4]. Our numerical result exhibits a maximum at the magnetic field value $B_p \approx 0.8B_{crit}^{Beton}$. This is in very good agreement with recent experiments [20], although the numerical factor 0.8 should depend on the specific form of the potential modulation. The corresponding results for a pure magnetic modulation look qualitatively very similar and are not shown.

The inset of Fig. 4 shows our numerical results for strong purely magnetic modulations of the form $B_m(x) = B_m^0 \cos qx$. The modulation strength is given in units of $1/qR_m = aeB_m^0/(2\pi mcv_F)$, $\Delta\rho_{xx}/\rho_0$ in units of $\omega_m^2\tau^2 = \lambda^2/R_m^2$. With increasing modulation strength (different curves in the inset of Fig. 4), the extent of the positive magnetoresistance regime at low B_0 values increases and $\Delta\rho_{xx}$ reaches a maximum at $B_p \approx 0.7B_m^0$ (note that $B_{crit} = B_m^0$, see appendix B). The magnetoresistivity at B_p increases with increasing modulation strength, and the low-field Weiss oscillations are successively suppressed (the fluctuations in the curve for $1/qR_m = 0.34$ are due to numerical errors, which become larger with increasing modulation strength). Again the dependence of the magnetoresistance on the modulation strength is very similar for the electric cosine modulation and will not be shown explicitly. For large mean free path and $V_0/E_F > 0.2$, fluctuations of the calculated curves indicate convergence problems. We found, however, that the relation $B_p \approx 0.8B_{crit}^{Beton}$, i.e. a linear increase of B_p with the modulation amplitude V_0, is reasonably well satisfied up to $V_0/E_F \approx 0.5$, in agreement with Ref. [4] and with a recent modification of Středa's semiclassical approach [13]. The peak values of the magnetoresistivity can be well fitted by

$$\Delta\rho_{xx}(B_p)/\rho_0 = 7.01\,(V_0/E_F)^2 + 54.7\,(V_0/E_F)^3\,. \quad (5.1)$$

Whereas in the regime of Weiss oscillations ($B_0 > 3B_{crit}$) $\Delta\rho_{xx}$ is proportional to $(V_0/E_F)^2$ and the analytic approximation (4.1) is valid, the peak value at B_p is dominated by the third order term in Eq. (5.1) for $V_0/E_F > 0.15$. For weak modulation ($V_0/E_F < 0.15$), $\lambda q \gg 1$ and $B_0 = 0$, our numerical results are consistent with the analytically calculated algebraic dependence $\Delta\rho_{xx}(0)/\rho_0 \propto (V_0/E_F)^{3/2}$ (see Appendix B). This indicates that, at low magnetic fields where channeled orbits exist, an expansion of the magnetoresistance in powers of the modulation strength is not possible. To check this, we expanded the Boltzmann equation (3.14) in powers of V_0/E_F and solved it up to the order $(V_0/E_F)^8$. In the regime around B_p the higher order contributions lead to high-frequency oscillations, but we found no indication of convergence or positive magnetoresistance.

C. Anisotropic scattering

Experiments in the regime of Weiss oscillations give evidence that the mobility of the 2DES is dominated by scattering of the electrons by remote ionized impurities, which should lead to predominant forward scattering. To investigate the effect of anisotropic scattering, we simulate the angular dependence of the differential scattering cross section in the collision operator, Eq. (2.10), by the simple model

$$P_p(\varphi) = c + (1-c)\,\frac{(p!)^2}{(2p)!}\left(2\cos\frac{\varphi}{2}\right)^{2p}\,, \quad (5.2)$$

where c controls a constant background ($0 \leq c \leq 1$), and p the sharpness of the forward scattering peak. This model is convenient, since it has simple Fourier coefficients, $\gamma_0 = 1$, $\gamma_m = (1-c)(p!)^2/[(p+m)!(p-m)!]$ for $|m| \leq p$, and $\gamma_m = 0$ for $|m| > p$ [cf. Eq. (2.11)]. According to Eq. (2.12), the ratio of transport time and relaxation time τ increases then with increasing p as $\tau_{tr}/\tau = (p+1)/(cp+1)$.

Keeping τ fixed, we have calculated magnetoresistivity curves for several values of the parameters c and p. We found as a general trend that, with inreasing τ_{tr}/τ, for both electric and magnetic modulations the values of $\Delta\rho_{xx}/\rho_0$ [with $\rho_0 = m/(e^2\bar{n}_{el}\tau_{tr})$] increase slightly faster than proportional to τ_{tr}/τ. On an absolute scale this means that the values of $\Delta\rho_{xx}/\rho_0$ change little near well developed minima but increase stronly at the peaks, resulting in a strong increase of the amplitudes of the Weiss oscillations. In Fig. 5 we have, for increasing τ_{tr}, reduced the relaxation time τ so that the maximum value of $\Delta\rho_{xx}/\rho_0$ near $1/qR = 0.26$ remains unchanged. In this plot higher-index Weiss oscillations are increasingly damped with increasing importance of small-angle scattering. Comparison with typical experiments shows that small-angle scattering is indeed important. Apart from the quantum mechanical Shubnikov-de Haas oscillations at large B_0, we could nicely fit experimental curves in the whole region $1/qR \leq 0.8$ with typical values of $\tau_{tr}/\tau \geq 2$.

D. Mobility modulation

Finally, we have considered, within the relaxation time approximation, a mixed electric modulation $V(x) = V_0 \cos qx$ and mobility modulation $1/\tau(x) = \bar{\tau}^{-1} + r_m \cos qx$. We choose an in-phase modulations, $V_0 \cdot r_m > 0$. We found that, for $V_0/E_F \ll 1$ and $\bar{\tau} v_F q \gg 1$, the effect of the mobility modulation on ρ_{xx} is rather small. This is understandable in the regime of Weiss oscillations, since, according to Fig. 1, $\Delta\rho_{xx}/\rho_0$ becomes independent of λq for $\lambda q \geq 25$. It is also expected for $B_0 = 0$ in the local limit, where $\Delta\rho_{xx}/\rho_0$ becomes proportional to $(V_0/E_F) \cdot \bar{\tau} r_m$ (For the corresponding step modulations one has $\bar{\rho}_{xx}/\rho_0 = [1 + (V_0/E_F)\bar{\tau} r_m]/[1 - (V_0/E_F)^2]$.)

In Fig. 6 we show the results for pure electric and pure mobility modulations as solid lines. Whereas the electric modulation enhances ρ_{xx}, the mobility modulation reduces ρ_{yy}. The Weiss oscillations of both resistivity components are in phase, as we have seen already from the analytical solution Eq. (4.1) - (4.3). For the pure mobility modulation, there is no positive magnetoresistance at small B_0 values, since there are no channeled orbits in this case.

Results for the case of mixed modulations are shown as broken lines. For the diagonal resistivities, the changes compared to the pure cases are essentially a positive offset of ρ_{xx} and a negative offset of ρ_{yy}, as expected from Eqs. (3.11) and (3.16). The enhancement of ρ_{xx} at small B_0 values ($R > R_{crit}$) is plausible, since now in the regime of channeled trajectories the mobility is increased. Therefore, the importance of these trajectories and thus the value of the magnetoresistivity is enhanced.

For the mixed case, we obtain also a correction to the Hall resistivity. To accommodate this correction in the same figure, we have subtracted the Drude value, which increases linearly with B_0. In the regime of Weiss oscillations, $\Delta\rho_{xy}$ is roughly proportional to the slope of ρ_{xx}/ρ_0 as function of $B_0 \propto 1/qR$.

VI. SUMMARY

Using the linearized stationary Boltzmann equation and including anisotropic scattering in the collision operator, we have investigated the effect of one-dimensional periodic modulations on the resistivity tensor of a two-dimensional electron system in a perpendicular magnetic field B_0. Even in the relaxation time approximation (for isotropic scattering) we have carefully considered the back-scattering term of the collision operator, which ensures relaxation of the distribution function towards its local equilibrium value in the stationary state, and not to the total equilibrium distribution. This is of principle importance for the modulated systems, since it ensures that the approximation does not violate the equation of continuity, and it has qualitative as well as quantitative consequences for the resistivity tensor.

The qualitative consequences concern the form of the resistivity tensor. In the absence of mobility modulations, electric and magnetic modulations in x-direction affect only ρ_{xx}, whereas the other components of the resistivity tensor remain those of the unmodulated system. This result, which is not obvious in the presence of a magnetic field, has first been obtained by Beenakker [10] for a pure electric modulation in the relaxation time approximation. In Appendix A we show that this result also holds for mixed electric and magnetic modulations and for anisotropic scattering. Moreover, we show that, in the absence of electric and magnetic modulations, a mobility modulation in x-direction affects only ρ_{yy}. In Sect. III A we have derived the same tensor structure in the local limit. This more transparent derivation elucidates the important role played by the continuity equation.

Simpler calculations based on an evaluation of the familiar velocity-velocity-correlation formula for the conductivity tensor are not in agreement with these exact results on the structure of the resistivity tensor [10,14,15], although they may yield reasonable approximate results under certain conditions.

For weak modulations, isotropic scattering, and sufficiently high magnetic field B_0, we followed an approximation scheme proposed by Beenakker [10] and obtained analytic results for the magnetoresistivity tensor (Sect. IV A). In the clean limit (mean free path much larger than modulation period) these results reduce to those

of the simpler calculations, apart from a characteristic denominator, which can be traced back to result from the back-scattering term in the collision operator. The absence of this denominator in the simplified treatment [14,15] indicates that this treatment does not correctly describe the relaxation towards the local equilibrium, and thus violates the equation of continuity. Quantitatively, the deviation of the denominator from unity is not large if the cyclotron radius R (of an electron at the Fermi energy) is larger than the period a of the modulation. Therefore, in the regime of Weiss oscillations the simplified treatment gives reasonable results. However, in the quasi-local regime of high magnetic fields ($R < a$), the denominator becomes small and the simplified treatment becomes wrong. In this regime the correct treatment of the equation of continuity is quantitatively important, see Sect. IV A 1.

The analytic approximation, which implicitly assumes weakly perturbed cyclotron motion, breaks down at low magnetic fields, where "channeled orbits" become important [4]. At $B_0 = 0$ and for $\lambda q \gg 1$, these should lead to an interesting algebraic dependence of the resistivity on the modulation strength, as we demonstrate in Appendix B. To include the effect of the channeled orbits, we have solved the linearized Boltzmann equation numerically. The numerical results in the relaxation time approximation (isotropic scattering) give a qualitatively correct description of the experimental result for ρ_{xx} in the range of small magnetic fields, where a pronounced positive magnetoresistance is obtained, and of intermediate magnetic fields, where the Weiss oscillations are observed. At $B_0 = 0$, they are also consistent with analytical results which are derived in Appendix B. A nearly quantitative description of the experimental ρ_{xx} [1] in the weak and intermediate magnetic field regime is obtained from a numerical calculation considering predominant forward scattering (Sect. V C).

In an attempt to find a classical explanation of the *anti-phase* Weiss oscillations observed in ρ_{yy} [1], we have also investigated a mobility modulation. We obtained, however, only weak *in-phase* oscillations, both analytically and numerically. Thus, this attempt of a classical explanation of the ρ_{yy} oscillations fails, whereas the quantum treatment provides a convincing explanation of these oscillations as a density-of-states effect [7].

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APPENDIX A: MOMENTS OF BOLTZMANN'S EQUATION

To investigate the structure of the tensor $\hat{\rho}$, we multiply the linearized Eq. (2.7) with $e^2 D_0 v(x)\mathbf{u}(\varphi)$ and average with respect to both φ and x over a full period. Integrating by parts eliminates the derivatives of the distribution function $\phi(x,\varphi)$, see Eq. (2.9), and the collision operator is evaluated using the real and imaginary part of

$$\int_{-\pi}^{\pi} \frac{d\varphi}{2\pi}\, e^{i\varphi} C[\phi; x, \varphi] = -\frac{1}{\tau_{tr}(x)} \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi}\, e^{i\varphi} \phi(x,\varphi)\,, \tag{A1}$$

where only the effective transport scattering rate $1/\tau_{tr}(x)$ enters [cf. Eq. (2.11)]. Multiplying with $\bar{\tau}_{tr}$ and rearranging terms we can write the result as

$$\hat{\rho}^D \langle \mathbf{j}(x) \rangle = \mathbf{E}^0 + \mathbf{\Delta}\,, \tag{A2}$$

where $\hat{\rho}^D$ is the Drude resistivity tensor of the homogeneous system, and [with $V'(x) = dV/dx$]

$$\Delta_x = -\langle [V'(x)/E_F]\phi^{loc}(x) + \rho_0 \bar{\tau}_{tr} \omega_m(x) j_y(x) \rangle\,, \tag{A3}$$

$$\Delta_y = -\rho_0 \bar{\tau}_{tr} \langle r_{tr}(x) j_y(x) \rangle\ . \tag{A4}$$

Here we have explicitly used that, according to the equation of continuity, $j_x(x) = \langle j_x(x) \rangle$ is spatially constant, and that the spatial average values of $\omega_m(x)$, $V(x)$, and $r_{tr}(x)$ vanish.

For the unmodulated system $\mathbf{\Delta} = \mathbf{0}$, and Eq. (A2) reduces to the well known Drude form of Ohm's law with homogeneous current density $\mathbf{j}(x) = \langle \mathbf{j}(x) \rangle = \mathbf{j}^0$. For the modulated system with the same $\mathbf{E}^0$, we may write $\langle \mathbf{j}(x) \rangle = \mathbf{j}^0 + \langle \delta\mathbf{j}(x) \rangle$, so that

$$\hat{\rho}^D \langle \delta\mathbf{j}(x) \rangle = \mathbf{\Delta}\,. \tag{A5}$$

To calculate these corrections, it is convenient to write the solution of Eq. (2.7) in the form

$$\phi(x,\varphi) = \phi^0(\varphi) v(x)/v_F + \rho_0 \tau_{tr} \chi(x,\varphi)\,, \tag{A6}$$

where $\phi^0(\varphi) = \rho_0 \tau_{tr} v_F \mathbf{u}(\varphi) \cdot \mathbf{j}^0$ is the corresponding solution for the unmodulated system. This yields

$$\mathbf{j}(x) = \mathbf{j}^0 v^2(x)/v_F^2 + \delta\mathbf{j}(x)\,, \tag{A7}$$

with

$$\delta\mathbf{j}(x) = \frac{2}{v_F^2} \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi}\, v(x)\mathbf{u}(\varphi)\chi(x,\varphi)\,, \tag{A8}$$

and $\chi(x,\varphi)$ satisfies Eq. (2.7) with a modified inhomogeneity which is linear in the modulation quantities $\omega_m(x)$, $V(x)$, and $r_{tr}(x)$,

$$\begin{aligned}[\mathcal{D} - C]\,\chi = (\,-\,v dv/dx + \omega_m v_y - r_{tr} v_x) j_x^0 \\ -\,(\omega_m v_x + r_{tr} v_y) j_y^0\,,\end{aligned} \tag{A9}$$

where $(v_x, v_y) = v(x)(\cos\varphi, \sin\varphi)$. We may write $\chi = \chi_1 + \chi_2$, where χ_2 solves

$$[\mathcal{D} - C]\,\chi_2 = -v(x)\cos\varphi\,[r_{tr} j_x^0 + \omega_m j_y^0]\,. \tag{A10}$$

The solution is independent of φ, and satisfies

$$d\chi_2/dx = -[r_{tr} j_x^0 + \omega_m j_y^0]\,. \tag{A11}$$

It does not contribute to the current density, Eq. (A8), but it contributes to the local equilibrium distribution, and thus contributes to the inhomogeneity of Eq. (A3),

$$\left\langle \frac{dV/dx}{E_F}\,\phi^{loc} \right\rangle = \rho_0 \bar{\tau}_{tr} \left\langle \frac{dV/dx}{E_F} \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi}\,\chi_1 + \frac{V}{E_F}\,[r_{tr} j_x^0 + \omega_m j_y^0] \right\rangle . \tag{A12}$$

From Eqs. (A7) to (A12) we obtain

$$\Delta_x = -\rho_0 \bar{\tau}_{tr} \left\langle r_{tr} \frac{V}{E_F} j_x^0 + \frac{2}{v_F^2} \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} \left(\frac{v_F^2}{2} \frac{dV/dx}{E_F} + \omega_m v_y \right) \chi_1 \right\rangle , \tag{A13}$$

$$\Delta_y = -\rho_0 \bar{\tau}_{tr} \left\langle -r_{tr} \frac{V}{E_F} j_y^0 + \frac{2}{v_F^2} \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi}\, r_{tr} v_y \chi_1 \right\rangle . \tag{A14}$$

Finally, we write $\chi_1 = \chi_x j_x^0 + \chi_y j_y^0$, where, according to Eqs. (A9) and (A10), χ_ν (for $\nu = x, y$) satisfies Eq. (3.14) with g_ν given by Eq. (3.13). We obtain

$$\mathbf{\Delta} = \rho_0 \hat{\mathbf{\Gamma}}\,\mathbf{j}^0\,, \tag{A15}$$

where $\hat{\mathbf{\Gamma}}$ is defined by Eq. (3.11).

From these results we can show that the magnetoresistivity tensor reveals a very simple structure in the two important special cases, where there is either no modulation of the scattering rate, or where only the scattering rate is modulated. Let us first assume that we have an electrical and a magnetic modulation, but no mobility modulation, $r(x) \equiv 0$. Then $\Gamma_{xx} = -G_{xx}$ and all other tensor components of $\hat{\mathbf{\Gamma}}$ vanish, so that $\Delta_x = \rho_0 \Gamma_{xx} j_x^0$, while $\Delta_y = 0$. Thus, according to Eq. (A5), $\langle \delta j_y \rangle = \omega_c \bar{\tau}_{tr} \langle \delta j_x \rangle$ and $\langle \delta j_x \rangle = \tilde{\Gamma}_{xx} j_x^0$, with $\tilde{\Gamma}_{xx} = \Gamma_{xx}/[1 + (\omega_c \tau_{tr})^2]$. If we now consider the effective conductivity tensor $\hat{\sigma} = \hat{\rho}^{-1}$ of the modulated system, $\mathbf{j}^0 + \langle \delta \mathbf{j} \rangle = \hat{\sigma} \mathbf{E}^0$, we obtain for the components of $\hat{\sigma}$ in terms of those of the Drude $\hat{\sigma}^D = [\hat{\rho}^D]^{-1}$: $\sigma_{\mu\nu} = \sigma^D_{\mu\nu}(1 + \tilde{\Gamma}_{xx})$ for $(\mu\nu) = (xx)$, (xy), and (yx), whereas $\sigma_{yy} = \sigma^D_{yy}[1 - (\omega_c \bar{\tau}_{tr})^2 \tilde{\Gamma}_{xx}]$. Tensor inversion shows that $\hat{\rho}$ differs from the Drude resistivity tensor $\hat{\rho}^D$ of the unmodulated system only in the xx component,

$$\Delta\rho_{xx}/\rho_0 = \rho_{xx}/\rho_0 - 1 = -\Gamma_{xx}/[1 + \tilde{\Gamma}_{xx}]\,, \tag{A16}$$

while all other components are not modified by the modulation. This result, first pointed out by Beenakker [10] for purely electrostatic modulation within the relaxation time approach, is thus shown to hold also in the presence of a magnetic modulation and anisotropic scattering.

The other simple case is that without electric and magnetic modulation, $V(x) \equiv 0$, $\omega_m(x) \equiv 0$, but with a finite mobility modulation $r_{tr}(x)$. In this case $\Delta_x = 0$ and $\Delta_y = \rho_0 \Gamma_{yy} j_y^0$. The same procedure yields now a resistivity tensor, which differs from that of the unmodulated system only in the yy component,

$$\Delta\rho_{yy}/\rho_0 = -\Gamma_{yy}/(1 + \Gamma_{yy}/[1 + (\omega_c \bar{\tau}_{tr})^2])\,. \tag{A17}$$

In the general case, with mobility modulation and electric or magnetic modulation, we obtain from Eqs. (3.2), (A2), and (A15) the effective conductivity tensor given by Eq. (3.10).

APPENDIX B: CHANNELED ORBITS

1. Critical field B_{crit}

For a smooth modulation, the condition for the existence of channeled orbits is apparently, that the functions $x_0^-(x)$ and $x_0^+(x)$ have local extrema, i.e., that the equation

$$1 + \frac{B_m(x)}{B_0} \mp \frac{R}{2} \frac{V'(x)/E_F}{\sqrt{1 - V(x)/E_F}} = 0 \tag{B1}$$

has a solution. For a pure magnetic modulation of the form $B_m(x) = B_m^0 \cos qx$, we obviously find $B_{crit} = B_m^0$. For $B_0 < B_{crit}$ the total magnetic field $B(x) = B_0 + B_m(x)$ changes sign, and channeled orbits exist close to lines with $B(x) = 0$ [21].

For a pure electric modulation of the form $V(x) = -V_0 \cos qx$, Eq. (B1) is equivalent with a quadratic equation for $\cos qx$, which has a solution only if $1 - (qR)^2 + [(qR)^2 V_0/2E_F]^2 > 0$. This yields for the critical cyclotron radius

$$\frac{2}{qR_{crit}} = \frac{V_0}{E_F} \left[\frac{2}{1 + \sqrt{1 - (V_0/E_F)^2}} \right]^{\frac{1}{2}} . \tag{B2}$$

For $V_0 \ll E_F$, we recover the result $B_{crit} \approx B_{crit}^{Beton}(V_0) = cqV_0/ev_F$, which has previously been obtained by Beton *et al.* [4] from the balance of the average Lorentz force and the maximum electrical force qV_0 due to the modulation. For stronger modulation, the B_{crit} obtained from Eq. (B2) increases faster than linearly with V_0 and reaches the value $B_{crit} = \sqrt{2}\,B_{crit}^{Beton}$ at $V_0 = E_F$. Then all trajectories are confined to a single period of the potential. For $B_0 > B_{crit}$, i.e., for $2R < a\sqrt{2}/\pi$, all trajectories are "drifting" orbits, with left turning points on $x_0^+(x)$ and right turning points on $x_0^-(x)$, whereas for $B_0 < B_{crit}$ there exist also "channeled" trajectories having all their turning points either on $x_0^+(x)$ or on $x_0^-(x)$.

For the step modulation $V(x) = -V_0\mathrm{sign}(\cos qx)$, $x_0^+(x)$ and $x_0^-(x)$ are piecewise linear functions with discontinuities at the walls of the potential wells, so that Eq. (B1) can not be used to determine B_{crit}. Two types of channeled orbits may exist near a local minimum of $x_0^-(x)$ (maximum of $x_0^+(x)$): traversing trajectories with turning points on opposite walls of a potential well, and skipping orbits which are reflected only by one of the walls. Traversing trajectories exist for sufficiently weak magnetic fields, $B_0 < B_{crit}^{\rm step}(V_0) = mcv_F/eR_{crit}^{\rm step}$ with

$$\frac{a}{2R_{crit}^{\rm step}} = \sqrt{1+V_0/E_F} - \sqrt{1-V_0/E_F}\,. \tag{B3}$$

Skipping orbits survive at arbitrarily high magnetic fields. Physically, this is a consequence of the fact that the singular electric field at a potential discontinuity never becomes negligible as compared to the Lorentz force due to B_0. For small modulation amplitude we find $B_{crit}^{\rm step}(V_0) = (2/\pi)B^{Beton}(V_0)$. This indicates that the linear dependence of the critical magnetic field on the (weak) modulation strength holds independently of the detailed shape of the potential modulation, which may, however, affect the numerical prefactor.

A characteristic feature of the channeled orbits is that, even at $B_0 \neq 0$, their velocity in y-direction, $v_y = \omega_c(x - x_0)$, does not change sign.

2. Number of channeled orbits

We now calculate the fraction of electrons occupying channeled orbits. The electron density $n_{el}(x;E_F) = D_0[E_F - V(x)]\theta(E_F - V(x))$, i.e., the area density of occupied states at x with energy below E_F, is proportional to the area $\pi v^2(x)$ of the circle with radius $v(x) = v_F[1 - V(x)/E_F]$ in $\mathbf{v}$-space. The areal density of states at x with energy at E is given by the local density of states $D(x;E) = dn_{el}(x;E)/dE = D_0$. Both, $n_{el}(x;E)$ and $D(x;E)$ are independent of the magnetic field B_0, and so are their spatial average values $\bar{n}_{el}(E)$ and $D(E)$. An electron trajectory through x, at $B_0 = 0$, is a channeled orbit, if the energy available for the motion in the x-direction is smaller than the height of the potential barrier, $E_F - (m/2)v_y^2 < V_{max}$, where V_{max} is the maximum value of $V(x)$. With $v_y = v(x)\sin\varphi$, this means $\sin^2\varphi > \sin^2\varphi_0(x;E_F)$, where

$$\varphi_0(x;E_F) = \arcsin\sqrt{[E_F - V_{max}]/[E_F - V(x)]}\,. \tag{B4}$$

Thus, the contribution of channeled orbits to the local density of states is $D_{co}(x;E) = D_0[1 - (2/\pi)\varphi_0(x;E)]$, and the density of electrons in channeled orbits (with energy below E_F) is $n_{co}(x;E_F) = D_0[E_F - V(x)]\theta(E_F - V(x))$ if $E_F < V_{max}$, and

$$n_{co}(x;E_F) = \frac{2D_0}{\pi}\Bigg[\sqrt{E_F - V_{max}}\,\sqrt{V_{max} - V(x)} + (E_F - V(x))\arcsin\sqrt{\frac{V_{max} - V(x)}{E_F - V(x)}}\,\Bigg]\,, \tag{B5}$$

if $E_F > V_{max}$. For fixed average electron density $\bar{n}_{el}$, the Fermi energy is independent of the modulation strength, if $V_0 < E_F = \bar{n}_{el}/D_0$. For $V_0 > E_F$ and $B_0 = 0$, all trajectories become channeled orbits. For the step potential $V(x) = -V_0\mathrm{sign}(\cos qx)$, channeled orbits exist only in the potential wells, and Eq. (B5) yields the area density of channeled orbits in the well with $V_{max} = V_0$ and $V(x) = -V_0$. For $V_0 > E_F$ the requirement of fixed average density $\bar{n}_{el}$ yields a constant value of $E_F + V_0 = 2\bar{n}_{el}/D_0$ in the wells, so that for $V_0 > E_F$ the situation is identical to that at $V_0 = E_F$, with vanishing electron density in the barrier regions. Since the local density of states is D_0, we normalize, for arbitrary V_0, the fraction of channeled orbits for the step modulation on the total density of states in the wells, $f_{co}^{\rm step}(B_0=0,V_0) = D_{co}^{\rm step}(x;E)/D_0$ (averaging also over the barrier regions, where no channeled orbits exist, would reduce $f_{co}^{\rm step}(B_0,V_0)$ by a factor of 2). For $B_0 = 0$ we obtain

$$f_{co}^{\rm step}(0,V_0) = \frac{2}{\pi}\arcsin\sqrt{\frac{2\,V_0}{E_F+V_0}}\,, \tag{B6}$$

which for small $\epsilon = V_0/E_F$ has the expansion

$$f_{co}^{\rm step} = \frac{2\sqrt{2\epsilon}}{\pi} - \frac{(\sqrt{2\epsilon}\,)^3}{6\pi} + O(\epsilon^{\,5/2})\,. \tag{B7}$$

This algebraic dependence on the modulation strength V_0 is not an artifact of the step model. For the cosine modulation $V(x) = -V_0\cos qx$ one obtains, with $f_{co}^{\cos}(B_0,V_0) = D_{co}^{\cos}(E_F)/D_0 = \langle D_{co}^{\cos}(x,E_F)\rangle/D_0$,

$$f_{co}^{\cos}(0,V_0) = \frac{2}{\pi}\int_0^a \frac{dx}{a}\arcsin\sqrt{\frac{V_0 - V(x)}{E_F - V(x)}}\,, \tag{B8}$$

with the expansion

$$f_{co}^{\cos} = \frac{4\sqrt{2\epsilon}}{\pi^2} + \frac{(\sqrt{2\epsilon}\,)^3}{9\pi^2} + O(\epsilon^{\,5/2}) \tag{B9}$$

for small amplitudes.

For the step modulation, also the B_0-dependence of $f_{co}(B_0,V_0)$ can be calculated analytically. We suppress the lengthy result for $0 < B_0 < B_{crit}^{\rm step}$ [cf. Eq. (B3)] and give only the result for $B_0 > B_{crit}^{\rm step}$,

$$f_{co}^{\rm step} = \frac{4R}{a}\left[\sqrt{2\epsilon} - \sqrt{1-\epsilon}\,\arcsin\sqrt{\frac{2\epsilon}{1+\epsilon}}\,\right]\,. \tag{B10}$$

For the cosine model at $0 < B_0 < B_{crit}^{\cos}$, we can calculate the positions and the values of the local extrema of $x_0^-(x)$ [and similar for $x_0^+(x)$] analytically, and also $D_{co}^{\cos}(x;E)$. Contributions to the average of $D_{co}^{\cos}(x;E)$ come from the x-interval between the location x_{max} of the local maximum and the value x_{upper} satisfying

$x_0^-(x_{upper}) = x_0^-(x_{max})$. The value x_{upper} and the integral of $D_{co}^{\cos}(x;E)$ between x_{max} and x_{upper} are calculated numerically. The results are plotted in Fig. 7 for both the cosine and the step model. Figure 7(a) shows the magnetic field dependence and (b) shows the dependence on the modulation strength of $f_{co}(B_0, V_0)$, for $V_0 \leq E_F$ in all cases. For the cosine model at a given value of the magnetic field B_0, channeled orbits exist if $V_0/E_F > (2/qR)[1-(qR)^{-2}]^{1/2}$. Their number increases with increasing V_0. For $V_0 \to E_F$, $f_{co}(B_0, V_0)$ approaches, with vertical slope, a finite value, which decreases with increasing B_0. No channeled orbits survive if B_0 becomes so large that $2/qR \geq \sqrt{2}$. For the step model, the given B_0-value sets a critical modulation strength $V_{crit}^{\rm step}$ given by $V_{crit}^{\rm step}/E_F = (\pi/qR)[1-(\pi/2qR)^2]^{1/2}$. For $V_0 < V_{crit}^{\rm step}$ only skipping orbits exist, whereas for $V_0 > V_{crit}^{\rm step}$ also traversing orbits contribute to $f_{co}(B_0, V_0)$, which leads to a change of the curvature of the curves in Fig. 7(b) at $V_0 = V_{crit}^{\rm step}$. With increasing V_0, $f_{co}(B_0, V_0)$ increases and reaches at $V_0 = E_F$, also with vertical slope, the value

$$f_{co}^{\rm step}(B_0, E_F) = \frac{2}{\pi}\left[\frac{\pi}{2} - \arcsin\frac{b}{\sqrt{2}} + \frac{b/\sqrt{2}}{1+\sqrt{1-b^2/2}}\right], \tag{B11}$$

if $b \equiv \pi/qR < \sqrt{2}$, and $f_{co}^{\rm step}(B_0, E_F) = (2/\pi)\sqrt{2}/b$ if $b > \sqrt{2}$.

With the normalization chosen in Fig. 7(a), one obtains for the B_0 dependence of f_{co} nearly the same curve for all values of the modulation strength, $0 < V_0 < E_F$. It can be shown analytically that, for weak modulation and to lowest order in V_0/E_F, a single curve should result for each of the models. Numerically we find this to be true for $V_0/E_F < 0.2$. For the step model $f_{co}^{\rm step}(B_{crit}^{\rm step}(V_0), V_0)/f_{co}^{\rm step}(0, V_0)$ approaches the values $2/3$ for $V_0 \to 0$ and $2/\pi$ for $V_0 \to E_F$.

The algebraic dependence of the number of channeled orbits on the modulation strength leads to a similar algebraic V_0 dependence of their contribution to the resistivity at $B_0 = 0$, as we demonstrate now.

3. Resistance at $B_0 = 0$

We now evaluate, for $B_0 = 0$, the classical velocity-velocity-correlation formula for the conductivity in the extreme nonlocal limit $\lambda q \to \infty$. In this limit we expect the errors due to the violation of the continuity equation to be small, as we discussed in section IV A 1. In the limit $B_0 = 0$, the constant of motion is $\langle v_y \rangle$, the average velocity in y-direction. The "drifting orbits" transform into orbits, which are unbound in x-direction and lead to a periodic velocity $\mathbf{v}(x+a) = \mathbf{v}(x)$, whereas the "channeled orbits" remain bound in x-direction within one period, and their velocity in x-direction, v_x, vanishes on average. Then the conductivity tensor reduces to [15]

$$\sigma_{\mu\nu} = \frac{\sigma_0}{v_F^2/2}\int_0^a \frac{dx_0}{a}\int_{-\pi}^{\pi}\frac{d\varphi}{2\pi} v_\mu(\varphi, x_0)\bar{v}_\nu(\varphi, x_0), \tag{B12}$$

where $\bar{v}_\nu(\varphi, x_0)$ is the average of a velocity component along the trajectory starting (for an arbitrary value of y_0) at (x_0, y_0) with velocity $\mathbf{v} = v_F(x_0)(\cos\varphi, \sin\varphi)$, where $v_F(x_0) = v_F[1 - V(x_0)/E_F]^{1/2}$.

First, we consider a pure electric modulation, which implies that v_y is constant along any trajectory. Since the modulation potential $V(x)$ vanishes on average, we see immediately that $\sigma_{yy} = \sigma_0$ and $\sigma_{xy} = 0$ have the same Drude values as the unmodulated system. According to Eq. (B12), the channeled orbits do not contribute to σ_{xx}. For the trajectories extended in x-direction, one has $\bar{v}_x(\varphi, x_0) = a/T(\varphi, x_0)$, where

$$T(\varphi, x_0) = \int_0^a \frac{dx}{\sqrt{v_F^2[1 - V(x)/E_F] - v_y^2(\varphi, x_0)}} \tag{B13}$$

is the time an electron needs to traverse one period of the modulation. Since $T(\varphi, x_0)$ is an even function of φ, Eq. (B12) yields also $\sigma_{yx} = 0$. From Eqs. (B12) and (B13) we obtain for the electric cosine modulation

$$\frac{\sigma_{xx}^{\cos}}{\sigma_0} = (1+\epsilon)\,\Phi\!\left(\frac{2\epsilon}{1+\epsilon}\right), \tag{B14}$$

where $\epsilon = V_0/E_F$,

$$\Phi(\eta) = \eta^{3/2}\int_\eta^1 dt\ [\,t^2\,\mathrm{K}(t)\,\sqrt{t-\eta}\,]^{-1} \tag{B15}$$

and $\mathrm{K}(t)$ is the complete elliptic integral of the first kind [22]. Apparently $\sigma_{xx}^{\cos} \to 0$ for $V_0 \to E_F$. For $\epsilon \to 0$ one obtains a power expansion in $\sqrt{\epsilon}$, the coefficients of which can be calculated numerically. To leading order in ϵ one obtains

$$\frac{\Delta\rho_{xx}^{\cos}}{\rho_0} = 0.69454\left(\frac{V_0}{E_F}\right)^{3/2} + O([V_0/E_F]^{5/2}). \tag{B16}$$

For the step model the corresponding results can be calculated analytically. The exact result for the conductivity is

$$\frac{\sigma_{xx}^{\rm step}}{\sigma_0} = 1 - \frac{3-\epsilon}{\pi}\sqrt{\frac{1-\epsilon}{2\epsilon}} + \frac{3-2\epsilon-5\epsilon^2}{2\pi\,\epsilon}\arcsin\sqrt{\frac{2\epsilon}{1+\epsilon}}, \tag{B17}$$

with the leading term of the resistance correction

$$\frac{\Delta\rho_{xx}^{step}}{\rho_0} = \frac{16}{15\pi}\left(\frac{2V_0}{E_F}\right)^{3/2} + O([V_0/E_F]^{5/2}). \tag{B18}$$

Obviously, the step modulation yields the same half-integer power dependence of the resistivity on the modulation strength. Moreover, if we choose the mean values

of the squared modulation potentials equal, i.e., replace V_0 in the step potential by $V_0/\sqrt{2}$, the right hand side of Eq. (B18) becomes $0.571\,(V_0/E_F)^{3/2}$, where the prefactor is comparable with that in Eq. (B16).

The $B_0 = 0$ conductivities for pure magnetic modulations can be calculated similarly. For the modulation $B_m(x) = B_m^0 \cos qx$ one obtains

$$\frac{\sigma_{xx}^{\cos}}{\sigma_0} = (1+\alpha)^2\,\Phi\!\left(\frac{4\alpha}{(1+\alpha)^2}\right), \qquad \text{(B19)}$$

where $\Phi(\eta)$ is given by Eq. (B15), and $\alpha = 1/qR_m = \omega_m/qv_F$. For $\alpha \to 0$ this yields a power expansion in $\sqrt{\alpha}$ similar to Eq. (B16),

$$\frac{\Delta\rho_{xx}^{\cos}}{\rho_0} = 1.9645\,(qR_m)^{-3/2} + O([qR_m]^{-2}), \qquad \text{(B20)}$$

but now the correction term is larger. In this collisionless limit, the conductivity vanishes if all trajectories become bound in x-direction. For the cosine modulation this happens at $1/qR_m = 1$. For the magnetic step modulation, $B_m(x) = B_m^0 \mathrm{sign}(\cos qx)$ the corresponding condition is $a/4R_m = 1$, which means that a strip of constant $B_m(x)$ just can accommodate a cyclotron orbit with diameter $2R_m$. As a function of $a/4R_m$, the conductivity for the magnetic step modulation shows a similar behavior as that for the magnetic cosine modulation as a function of $1/qR_m$.

It can be shown analytically that, for $B_0 = 0$, $\sigma_{yy} = \sigma_0$ is not changed by the modulation. It is, however, interesting to calculate the fraction $\Delta\sigma_{yy}^{co}/\sigma_0$ contributed by the channeled orbits to σ_{yy}. This is easily shown to equal the fraction contributed to the density,

$$\frac{\Delta\sigma_{yy}^{co}}{\sigma_0} = \int_0^a \frac{dx}{a}\,\frac{n_{co}(x;E_F)}{\bar{n}_{el}}\,. \qquad \text{(B21)}$$

[1] D. Weiss, K. v. Klitzing, K. Ploog, and G. Weimann, Europhys. Lett. **8**, 179 (1989), see also in *High Magnetic Fields in Semiconductor Physics II*, edited by G. Landwehr, Springer Series in Solid-State Sciences Vol. **87** (Springer-Verlag, Berlin 1989), p. 357.
[2] R. R. Gerhardts, D. Weiss, and K. v. Klitzing, Phys. Rev. Lett. **62**, 1173 (1989).
[3] R. W. Winkler, J. P. Kotthaus, and K. Ploog, Phys. Rev. Lett. **62**, 1177 (1989).
[4] P. H. Beton *et al.*, Phys. Rev. B **42**, 9229 (1990).
[5] H. A. Carmona *et al.*, Phys. Rev. Lett. **74**, 3009 (1995).
[6] P. D. Ye *et al.*, Phys. Rev. Lett. **74**, 3013 (1995).
[7] C. Zhang and R. R. Gerhardts, Phys. Rev. B **41**, 12850 (1990).
[8] F. M. Peeters and P. Vasilopoulos, Phys. Rev. B **47**, 1466 (1993).
[9] D. Pfannkuche and R. R. Gerhardts, Phys. Rev. B **46**, 12606 (1992).
[10] C. W. J. Beenakker, Phys. Rev. Lett. **62**, 2020 (1989).
[11] P. Středa and A. H. MacDonald, Phys. Rev. B **41**, 11892 (1990).
[12] P. Středa, J. Kučera, and J. van de Konijnenberg, Physica Scripta **T39**, 162 (1991).
[13] J. Kučera, P. Středa, and R. R. Gerhardts, Phys. Rev. B **55**, 14439 (1997).
[14] R. R. Gerhardts, Phys. Rev. B **45**, 3449 (1992).
[15] R. R. Gerhardts, Phys. Rev. B **53**, 11064 (1996).
[16] J. M. Ziman, in *Electrons and Phonons* (Clarendon Press, Oxford, 1960).
[17] N. D. Mermin, Phys. Rev. B **1**, 2362 (1970).
[18] B. Y.-K. Hu, K. Sarker, and J. Wilkins, Phys. Rev. B **39**, 8468 (1989).
[19] H. Jou and A. M. Kriman, Superlattices and Microstructures **19**, 203 (1996).
[20] A. Soibel, U. Meirav, D. Mahalu, and H. Shtrikman, Phys. Rev. B **55**, 4482 (1997).
[21] J. E. Müller, Phys. Rev. Lett. **68**, 385 (1992).
[22] M. Abramowitz and I. A. Stegun, in *Handbook of mathematical functions* (Dover Publications, New York, 1968).

FIG. 1. (a) $S_q^{(0,0)}$ vs. $1/qR \propto B_0$ for different values of the mean free path λ, given in terms of the modulation period $a = 2\pi/q$; (b) and (c): selected magnetoresistance results for pure electric modulation $V(x) = V_0 \cos qx$ to order V_0^2, calculated from Eq. (4.1). In (b) $\Delta\rho_{xx}$ is plotted in units of $\rho_1 = (\rho_0/2)(V_0/E_F)^2$, and the dash-dotted line represents the local limit for $\lambda q = 1.0$. In (c) $\Delta\rho_{xx}$ is plotted in units of $\rho_2 = \rho_0(V_0/E_F)^2(\lambda q/2)^2$, and the thin line represents the approximation of Ref. [15].

FIG. 2. Magnetoresistivity vs. $1/qR \propto B_0$ for (a) mixed electric and magnetic, and (b) mixed electric and mobility modulations, calculated from Eqs. (4.1)-(4.4) for cosine modulations $V(x) = V_0 \cos qx$, $\omega_m(x) = \omega_m^0 \cos qx$, and $r(x) = r_m^0 \cos qx$, of comparable effective amplitudes $\alpha_V \equiv \lambda q V_0/2E_F$, $\alpha_B \equiv \tau\omega_m^0$ and $\alpha_\mu \equiv \tau r_m^0$, respectively, with $\lambda q = 30$. γ is a small, but otherwise arbitrary, dimensionless constant.

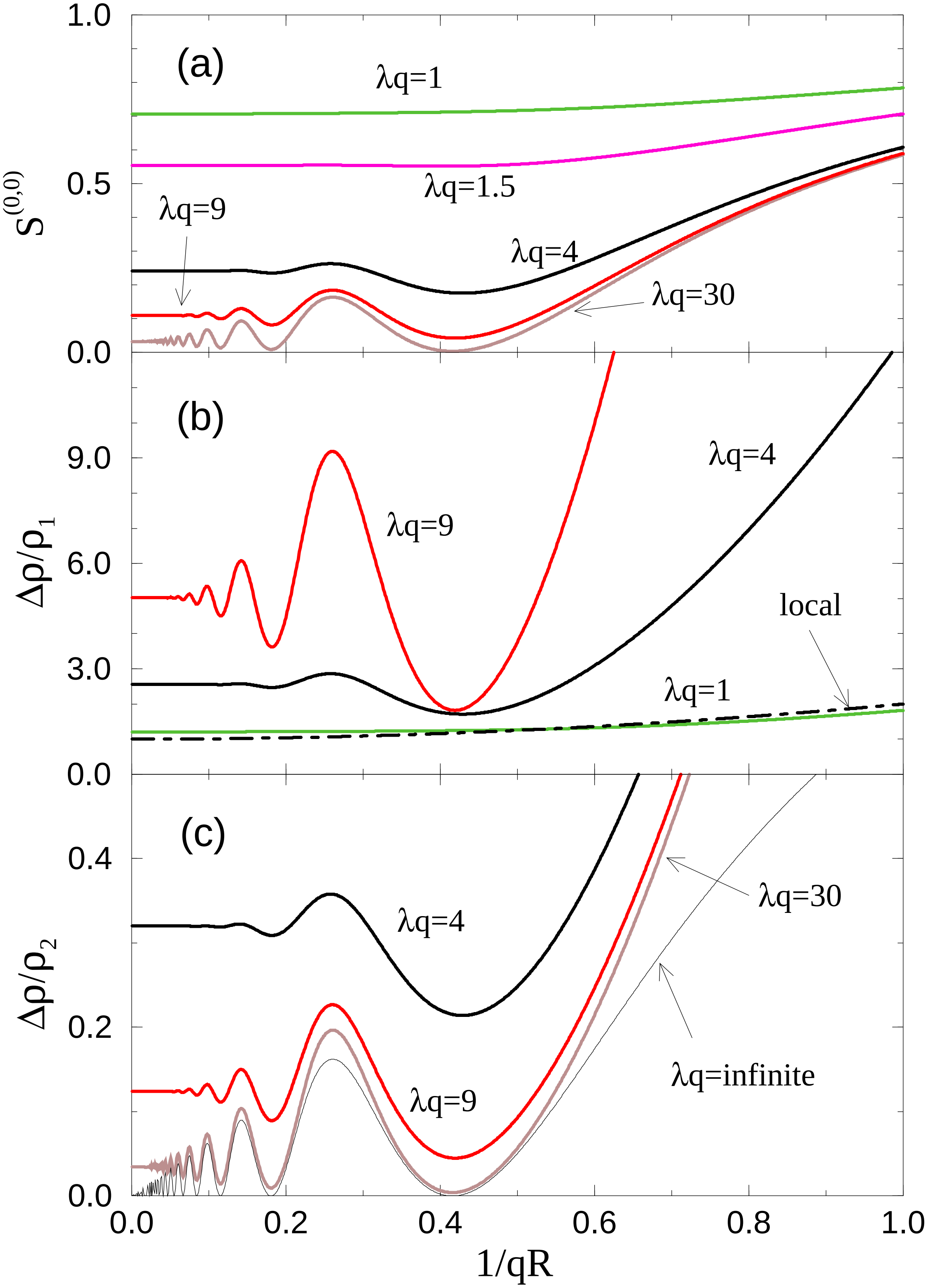
FIG. 3. Typical orbits (lower panel) in the electric superlattice potential $V(x) = 0.4\, E_F \cos qx$, with $q = 2\pi/a$, for three values of the magnetic field B_0 corresponding to $\beta = 1/qR = 0.25$, 0.1, and 0.001. The corresponding values of the constant of motion x_0 are indicated in the upper panels by horizontal bars ending on the curves $x_0^{\pm}(x)$ defining the turning points. Channeled orbits exist for $\beta > \beta_{crit} \approx 0.2$.

FIG. 4. Magnetoresistivity versus $1/qR \propto B_0$ for a weak electric cosine modulation with V_0/E_F=0.02 and λq=27.7. Solid line: numerical result, dotted line: analytical approximation (4.1), dash-dotted: contribution of channeled orbits (see text). Inset: the same for magnetic cosine modulations with $1/qR_m = 0.34$, 0.15, and 0.05 (from top to bottom at arrow, for further explanations see text) and λq=27.7.

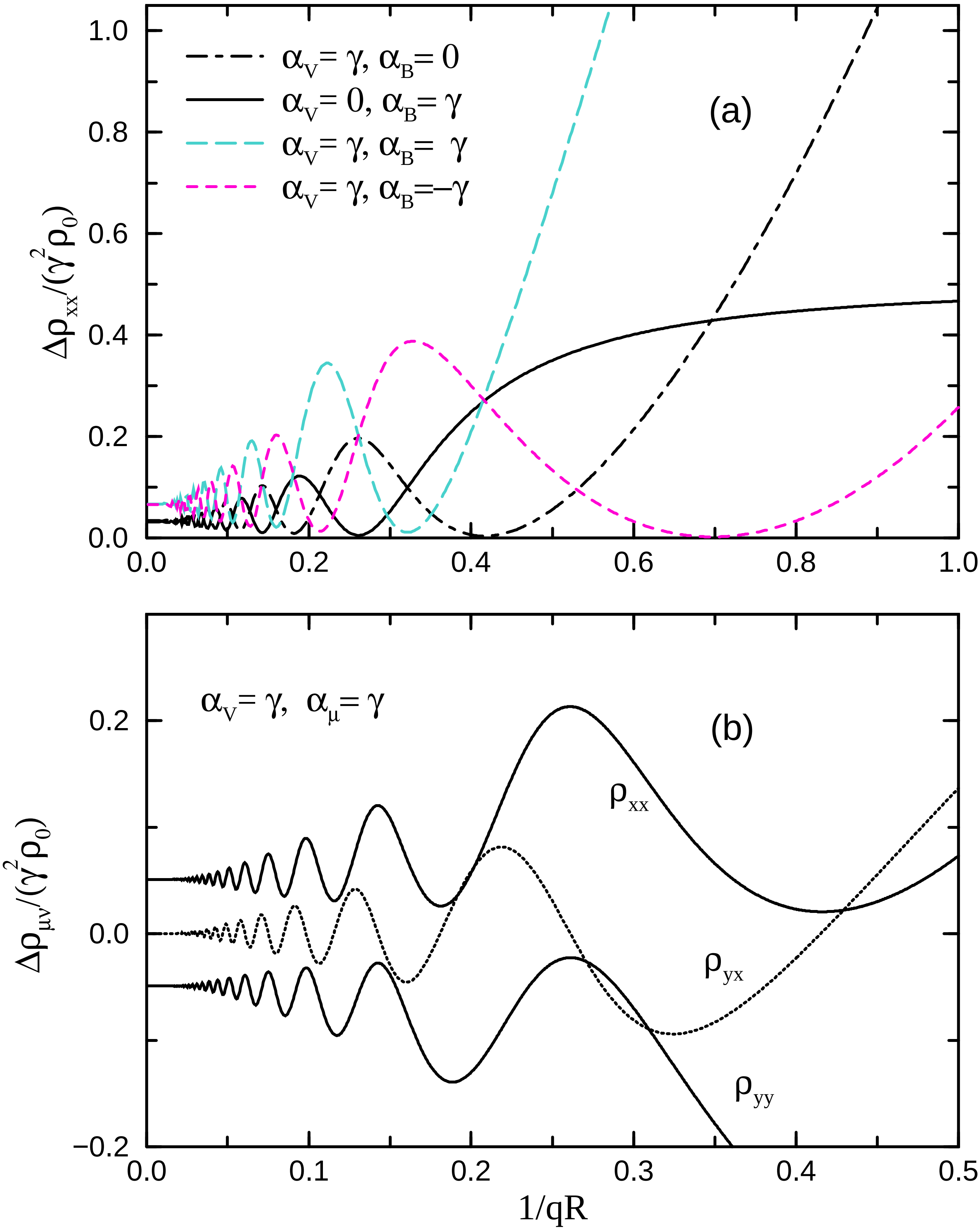
FIG. 5. Magnetoresistivity vs. $1/qR \propto B_0$ for anisotropic scattering, modelled by Eq. (5.2), and an electric cosine modulation with $V_0/E_F = 0.02$; $\tau_0 = 27.7/qv_F$.

FIG. 6. Magnetoresistivity vs. $1/qR$ for cosine modulation of the electric potential and the mobility, with $\bar{\tau} v_F q = 27.7$. Solid lines: ρ_{xx}/ρ_0 for pure electric modulation with $V_0/E_F = 0.2$ and ρ_{yy}/ρ_0 for pure mobility modulation with $r_m = 1/\bar{\tau}$; broken lines: mixed modulation.

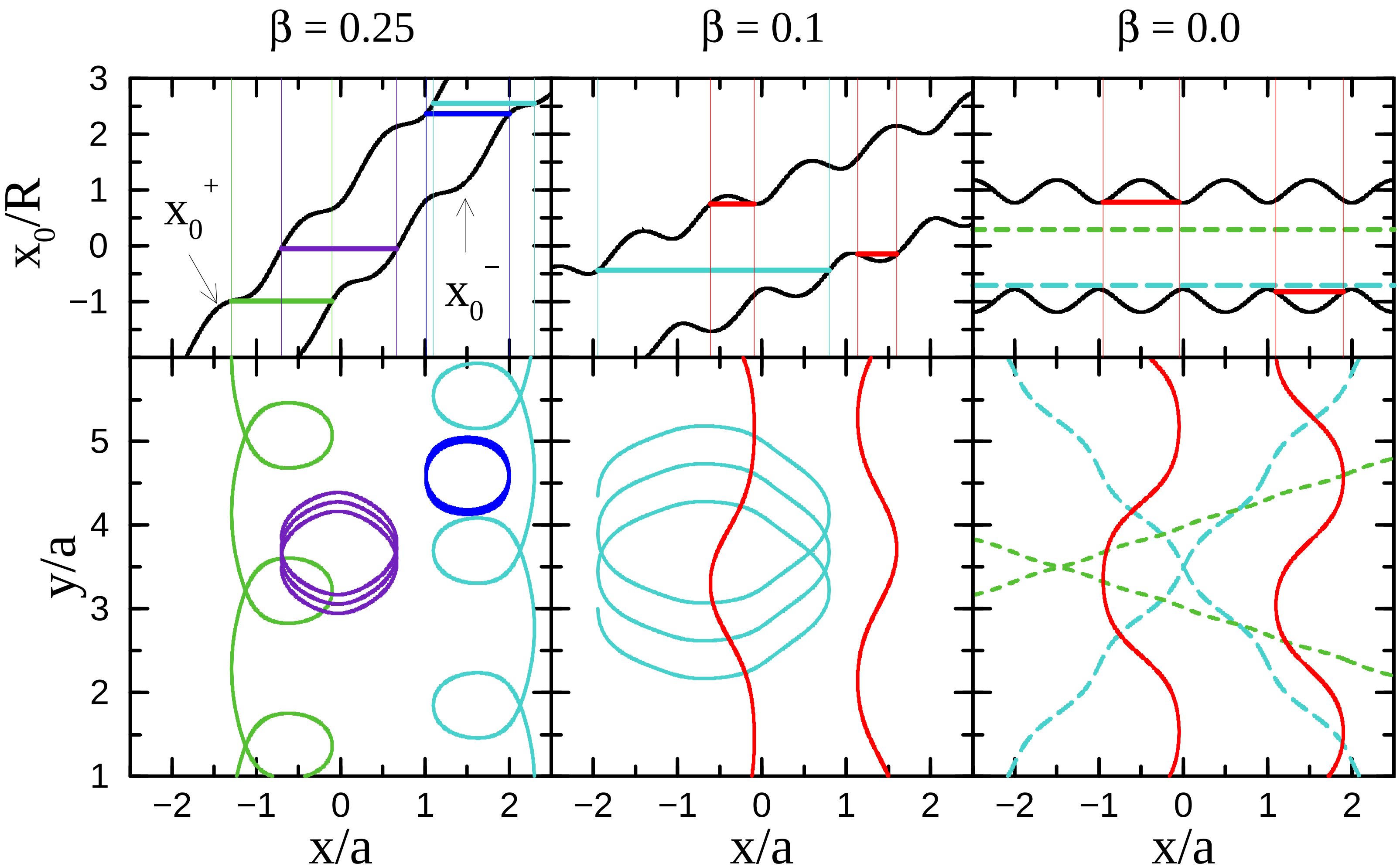
FIG. 7. Fraction of channeled orbits, as function of magnetic field B (a) and modulation strength V_0 (b), for an electric cosine modulation $V(x) = -V_0 \cos qx$ and a step modulation $V(x) = -V_0\, \mathrm{sign}(\cos qx)$; (a): B-dependence for fixed $V_0/E_F = 0.01$ (solid lines) and 0.95 (dash-dotted lines), (b): V_0-dependence for fixed $1/qR = 0.0$, 0.05, 0.2, and 0.4 (from top to bottom), with solid lines for the cosine and dash-dotted lines for the step modulation.



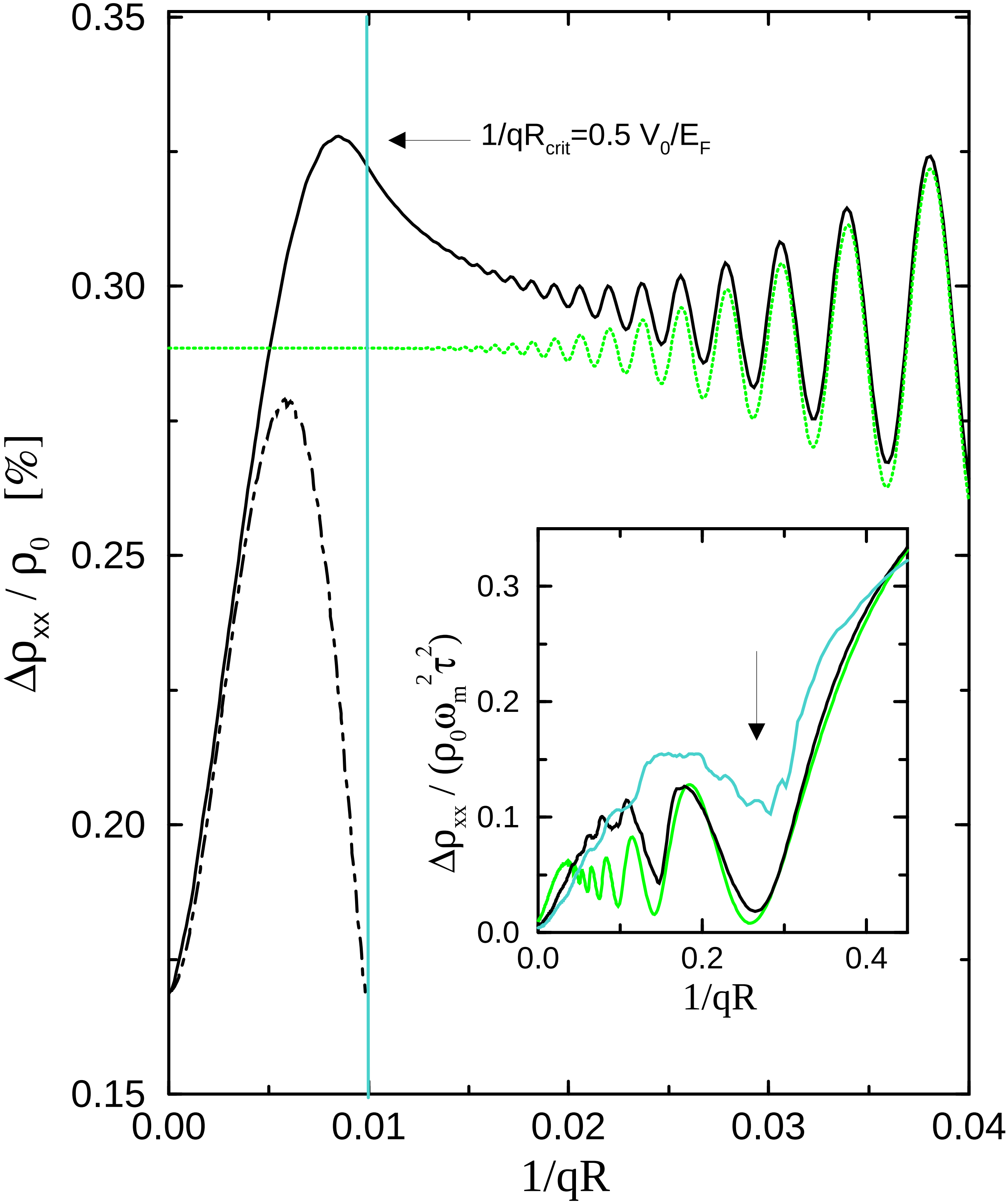
R.Menne, Figure 1



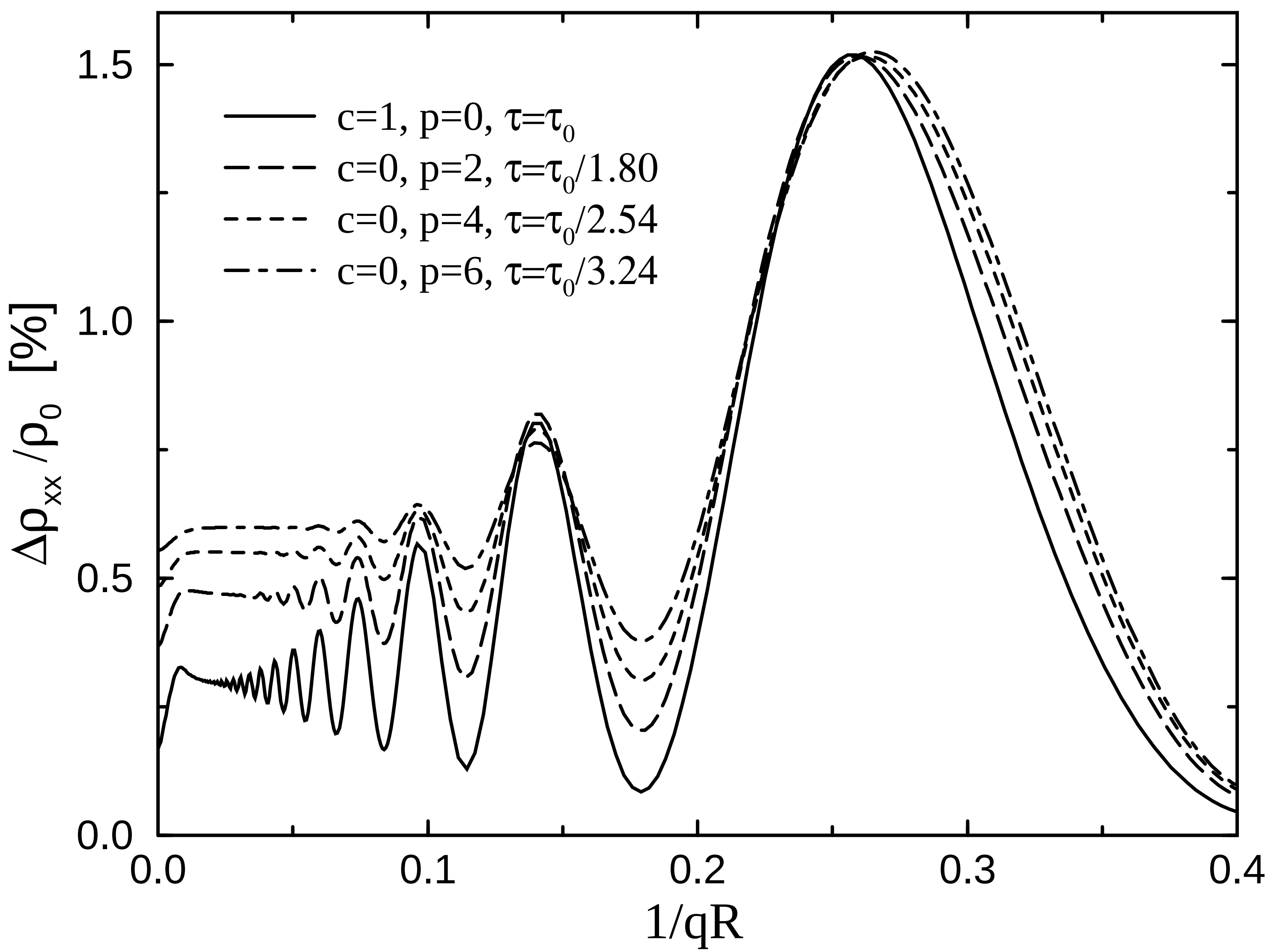
R.Menne, Figure 2



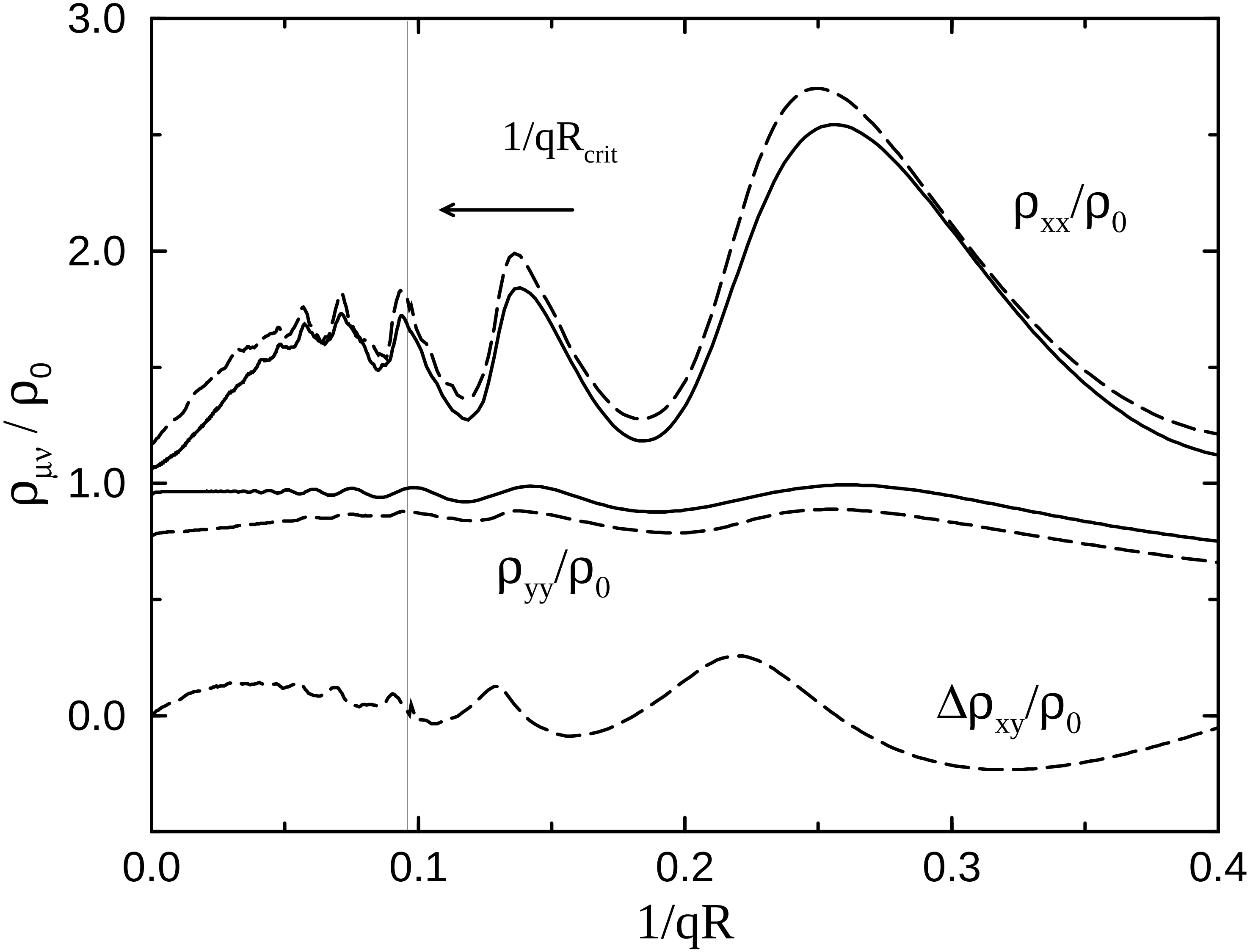
R.Menne, Figure 3



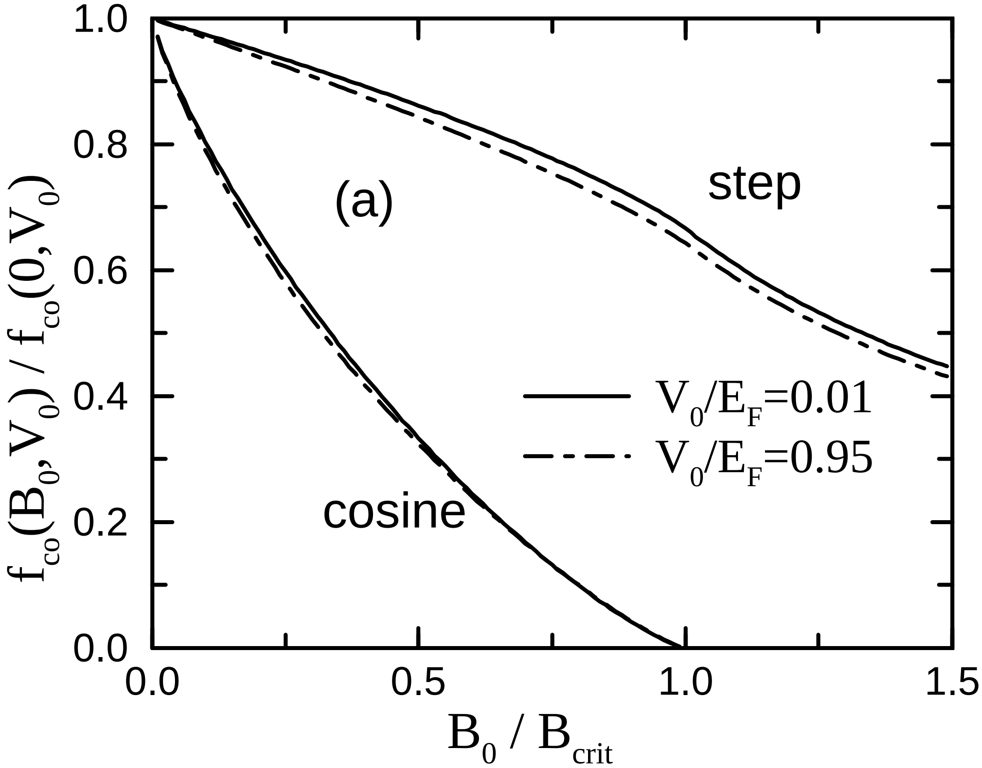
R.Menne, Figure 4

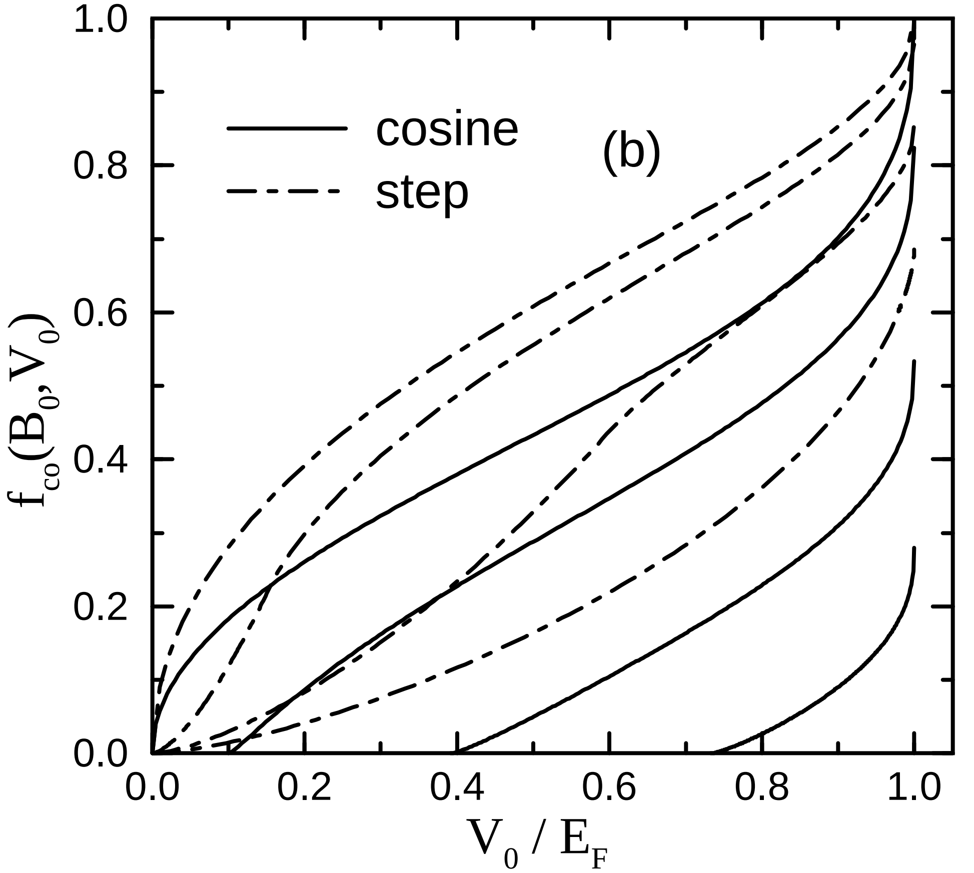


R.Menne, Figure 5



R.Menne, Figure 6





R.Menne, Figure 7